Introduction to Chiral Perturbation Theory

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Summary of last week

- Quark masses break chiral symmetries, but
  an effective low-energy theory (ChPT) can still be set up for small quark masses

- For small m_q
  - there is still a mass gap between the pseudo NGBs and the rest of the spectrum
  - the interaction between the pseudo NGBs is proportional to the 4-momentum p_μ and the quark masses m_q

- The effective theory (ChPT)
  - only contains the pseudo NGBs as active degrees of freedom
  - is a perturbative expansion in powers of p_μ and m_q
  - reproduces the symmetries of QCD, i.e. the chiral Ward identities

Summary of last week

- Leading order (LO) chiral Lagrangian

$$\mathcal{L}_2 = \frac{f^2}{4}\mathrm{tr}\,[\partial_\mu U \partial^\mu U^\dagger] + \frac{f^2 B}{2}\mathrm{tr}\,[MU^\dagger + UM^\dagger]$$

  contains

  - a term with two derivatives, $\mathrm{O}(p^2)$
  - a term with one quark mass, $\mathrm{O}(m_q)$ (recall $p^2 = M^2 \propto m_q$)
  - two low-energy coefficients (LECs) f, B not determined by chiral symmetry
  - the external source fields v, a if $\partial_\mu \to \nabla_\mu$

- Next-to-leading order (NLO) *Gasser-Leutwyler* Lagrangian contains

  - the terms of $\mathrm{O}(p^4, p^2 m_q\, m_q^2)$
  - has 8 LECs not determined by chiral symmetry
    10 + 2 including source fields for SU(3), 7 + 3 for SU(2) ChPT

Outline Part 3

- ChPT to one loop
- The Gasser-Leutwyler coefficients
- Systematics of the chiral expansion (Weinberg's power counting theorem)
- Some loose ends

ChPT to one loop

Loops

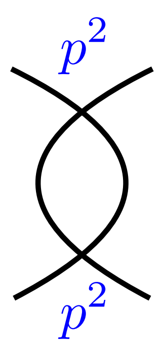
Why loops?

- Tree level scattering amplitudes computed with $\mathscr{L}_2 + \mathscr{L}_4$ are still real

  We need an imaginary part (optical theorem) to guarantee unitarity

  ➥ We need loop contributions with vertices from the LO Lagrangian $\mathscr{L}_2$

- The loop contribution with $\mathscr{L}_2$ vertices also contributes at $\mathrm{O}(p^4)$



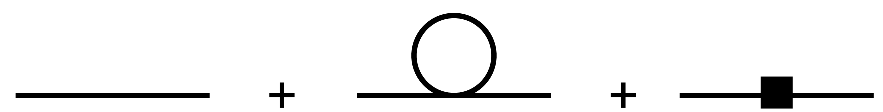
$$\Rightarrow \qquad p^4 \ln \frac{p^2}{\Lambda^2}$$

Example: The pion mass to 1-loop

Explicit example: 1-loop calculation of the pion mass

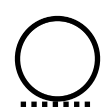
For simplicity:
- Consider SU(2) ChPT
- Assume isospin symmetry: $m_u = m_d = m$
  - ➥ 3 mass degenerate pions with tree level mass $M_0^2 = 2Bm$

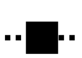
Consider $\tau(x,y) \equiv \langle 0 | T \pi_3(x) \pi_3(y) | 0 \rangle$

=  + ...

➥ $$\tau(p) = \mathrm{FT}[\tau(x,y)] = \frac{i}{p^2 - M_0^2 - \Delta M^2} \sim \frac{iZ_\pi}{p^2 - M_\pi^2}$$

$$-i\Delta M^2 = -i\Delta M^2_{\mathrm{loop}} - i\Delta M^2_{\mathrm{tree}}$$





Example: The pion mass to 1-loop

Step 1: Interaction vertices, expand $\mathcal{L}_2$ in pion fields

$$i\mathcal{L}_{2,4\pi} = \frac{i}{6f^2}\Big(-\partial_\mu\pi_3\partial^\mu\pi_3(\pi_1\pi_1+\pi_2\pi_2) - \pi_3\pi_3(\partial_\mu\pi_1\partial^\mu\pi_1+\partial_\mu\pi_2\partial^\mu\pi_2)\Big)\dots$$

$$-4\,p^2\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-M_0^2+i\epsilon} \qquad -4\int\frac{d^4k}{(2\pi)^4}\,\frac{i\,k^2}{k^2-M_0^2+i\epsilon}$$

$$\equiv \tilde{J}_1(M_0) \qquad\qquad \equiv \tilde{J}_2(M_0)$$

$$+\frac{iM_0^2}{24f^2}\Big(\pi_3^4+2\pi_3^2(\pi_1^2+\pi_2^2)\Big)+\dots$$

$$20\tilde{J}_1(M_0)$$

$$-i\Delta M^2_{\text{loop}} = \frac{i}{6f^2}\Big((-4p^2+5M_0^2)\tilde{J}_1(M_0)-4\tilde{J}_2(M_0)\Big)$$

Example: The pion mass to 1-loop

Step 2: Dimensional regularisation

$$\tilde{J}_1(M_0) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - M_0^2 + i\epsilon} \xrightarrow{\text{D dim, Wick}} A_0(M_0) = -\frac{M_0^2}{16\pi^2}\left(\Delta + 1 - \ln M_0^2\right)$$

$$\Delta = \frac{2}{\epsilon} - \gamma + \ln 4\pi$$

$$\tilde{J}_2(M_0) = \int \frac{d^4k}{(2\pi)^4} \frac{i\,k^2}{k^2 - M_0^2 + i\epsilon} \xrightarrow{\text{D dim, Wick}} M_0^2\, A_0(M_0) \qquad \epsilon = 4 - D$$

➥ $$\Delta M^2_{\text{loop}} = \frac{1}{96\pi^2}\left(-4\frac{p^2 M_0^2}{f^2} + \frac{M_0^4}{f^2}\right)(\Delta + 1 - \ln M_0^2)$$

Example: The pion mass to 1-loop

Step 3: The $\mathcal{L}_4$ tree contribution

Expand $\mathcal{L}_4$ in pion fields and collect terms quadratic in π_3

e.g. $\mathcal{L}_{4,L_6} = L_6\left[\mathrm{tr}(MU^\dagger + UM)\right]^2 \;=\; -\frac{1}{f^2}16L_6M_0^4\pi_3^2 + \ldots$

$$\Delta M^2_{\rm tree} = -8L_{45}\frac{p^2M_0^2}{f^2} + 16L_{68}\frac{M_0^4}{f^2}$$

$$L_{45} = 2L_4 + L_5$$
$$L_{68} = 2L_6 + L_8$$

➥ $$\Delta M^2 = \Delta M^2_{\rm loop} + \Delta M^2_{\rm tree} \;=\; \frac{1}{96\pi^2}\Big(-4\frac{p^2M_0^2}{f^2} + \frac{M_0^4}{f^2}\Big)\left(\Delta + 1 - \ln M_0^2\right)\ldots$$
$$\ldots - 8L_{45}\frac{p^2M_0^2}{f^2} + 16L_{68}\frac{M_0^4}{f^2}$$

Example: The pion mass to 1-loop

Step 4: Renormalisation

Introduce renormalised pion fields and GL coefficients and absorb the infinities:

$$\pi_3 = Z_\pi^{1/2}\,\pi_3^r$$

$$L_k = L_k^r(\mu) - \frac{\gamma_k}{32\pi^2}\left(\Delta + 1 - \ln\mu^2\right)$$

The γ_k are tuned $\quad \gamma_4 = \frac{1}{8} \quad \gamma_5 = \frac{1}{4} \quad \gamma_6 = \frac{3}{32} \quad \gamma_8 = 0$

such that the renormalised 2-pt function $\tau^r(p) = Z_\pi^{-1}\tau(p) \sim \frac{i}{p^2 - M_\pi^2}$ is finite

with finite pole mass $\quad M_\pi^2 = M_0^2\left(1 + \frac{M_0^2}{32\pi^2 f^2}\ln\frac{M_0^2}{\mu^2} - \frac{8M_0^2}{f^2}\left[L_{45}^r(\mu) - 2L_{68}^r(\mu)\right]\right)$

Exercise: reproduce this result
(starting from slide 10)

Example: The pion mass to 1-loop

Final result:

$$M_\pi^2 = M_0^2\left(1 + \frac{M_0^2}{32\pi^2 f^2}\ln\frac{M_0^2}{\mu^2} - \frac{8M_0^2}{f^2}[L_{45}^r(\mu) - 2L_{68}^r(\mu)]\right)$$

$$= M_0^2\left(1 + \frac{M_0^2}{2(4\pi f)^2}\ln\frac{M_0^2}{\Lambda_3^2}\right)$$

Rewriting:

$$8[L_{45}^r(\mu) - 2L_{68}^r(\mu)] \equiv \frac{1}{32\pi^2}\ln\frac{\Lambda_3^2}{\mu^2}$$

- Well-behaved finite result

  But: The necessary counter terms stem from $\mathcal{L}_4$ and not from $\mathcal{L}_2$

  ☞ expected from a non-renormalisable theory

- "*Non-renormalisability does not mean non-calculability*" J. Gasser, hep-ph/0312367

  Non-renormalisability manifests in new and increasing number of additional couplings

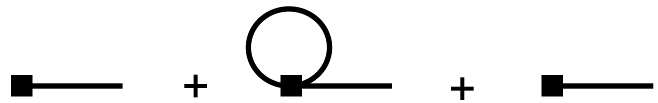
  2 in $\mathcal{L}_2$, 10 in $\mathcal{L}_4$, ...

- Note: μ dependence in $L^r(\mu)$ compensates the μ dependence in the "*chiral logarithm*"

- Note: Loop and $\mathcal{L}_4$ contribution are suppressed by $\dfrac{M_0^2}{(4\pi f)^2}$

Example: Pion decay constant to 1-loop

1-loop result for the pion decay constant

$\langle 0|A_\mu(0)|\pi^a(\vec{p})\rangle \cong$  also needed: Z_π

$$\Rightarrow \quad f_\pi = f\Big(1 - \frac{M_0^2}{16\pi^2 f^2}\ln\frac{M_0^2}{\mu^2} - \frac{8M_0^2}{f^2}L_{45}^r(\mu)\Big)$$

$f = f_{\pi,\mathrm{m}=0}$ chiral limit value

$$= f\Big(1 - \frac{M_0^2}{(4\pi f)^2}\ln\frac{M_0^2}{\Lambda_4^2}\Big)$$

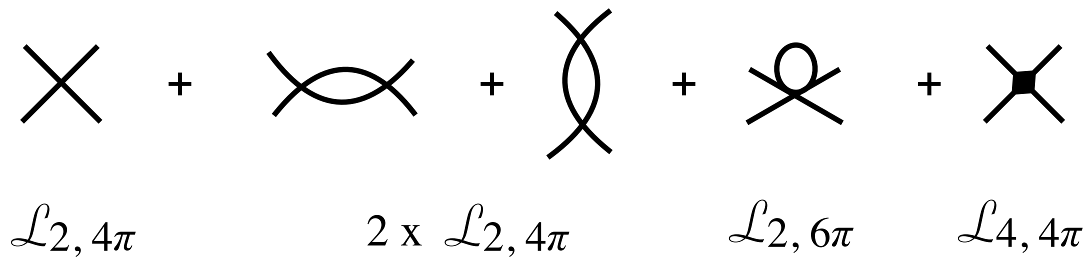
Rewriting $\quad \Lambda_4 = \Lambda_4(L_4^r, L_5^r)$

- Finite result
- Loop and $\mathcal{L}_4$ contribution suppressed by $\dfrac{M_0^2}{(4\pi f)^2}$

Example: $\pi\pi$ scattering to 1-loop

1-loop result for scattering amplitude for $\pi^+(p_1)\ \pi^+(p_2) \longrightarrow \pi^+(p_1')\ \pi^+(p_2')$

➥ scat. amplitude $T =$



$+\ Z_\pi$

➥ Threshold value, i.e. for $s = 4M_\pi^2, t = u = 0$ cp. with Lect 2, slide 28

$$T|_{\rm thr} = -\frac{2M_0^2}{f^2}\left(1 - \frac{4}{3}\frac{M_0^2}{(4\pi f)^2}\ln\frac{M_0^2}{\Lambda_1^2}\right) \qquad \Lambda_1 = \Lambda_1(L_1^r, L_2^r)$$

- Finite result
- Loop and $\mathcal{L}_4$ contribution suppressed by $\dfrac{M_0^2}{(4\pi f)^2}$

General observation in 1-loop results

- The observations hold in general
  The 1-loop and the $\mathcal{L}_4$ contributions are suppressed by relative to the $\mathcal{L}_2$ tree contributions
  $$\frac{p^2}{(4\pi f)^2} \qquad \frac{M_0^2}{(4\pi f)^2}$$
  - $\mathcal{L}_2$ tree contributions: *Leading order (LO)*
  - 1-loop and $\mathcal{L}_4$ contributions: *Next-to-leading order (NLO)*
- The expansion parameters are dimensionless with the chiral scale $\Lambda_\chi = 4\pi f \simeq 1\text{GeV}$
- In SU(3) ChPT we have also have the expansion parameter
  $$\frac{M_{K,\text{LO}}^2}{(4\pi f)^2} \qquad \frac{M_{\eta,\text{LO}}^2}{(4\pi f)^2}$$
  $M_\pi \approx 140$ MeV, $M_K \approx 500$ MeV, $M_\eta \approx 550$ MeV

➥ $$\frac{M_\pi^2}{(4\pi f)^2} \simeq 0.02 \qquad \frac{M_K^2}{(4\pi f)^2} \simeq 0.2 \qquad \frac{M_\eta^2}{(4\pi f)^2} \simeq 0.24$$

very small   reasonably small

The Gasser-Leutwyler coefficients

Example: Getting the GL coefficient L_5

1-loop results for decay constants in SU(3) ChPT:

$$\mu_P \equiv \frac{M_P^2}{32\pi^2 f^2} \ln \frac{M_P^2}{\mu^2}$$

$$f_\pi = f\left(1 - 2\mu_\pi - \mu_K + \frac{4M_\pi^2}{f^2} L_5^r(\mu) + \frac{8M_k^2 + 4M_\pi^2}{f^2} L_4^r(\mu)\right)$$

$$f_K = f\left(1 - \frac{3}{4}\mu_\pi - \frac{3}{2}\mu_K - \frac{3}{4}\mu_\eta + \frac{4M_K^2}{f^2} L_5^r(\mu) + \frac{8M_k^2 + 4M_\pi^2}{f^2} L_4^r(\mu)\right)$$

$$f_\eta = f\left(1 - 3\mu_K + \frac{4M_\eta^2}{f^2} L_5^r(\mu) + \frac{8M_k^2 + 4M_\pi^2}{f^2} L_4^r(\mu)\right)$$

$$\hookrightarrow \quad \frac{f_K}{f_\pi} = \left(1 + \frac{5}{2}\mu_\pi - \frac{1}{2}\mu_K - \frac{3}{4}\mu_\eta + \frac{4(M_K^2 - M_\pi^2)}{f^2} L_5^r(\mu)\right)$$

$$\left.\frac{f_K}{f_\pi}\right|_{\exp} = 1.22 \pm 0.01 \quad \Rightarrow \quad \begin{array}{l} L_5^r(M_\rho) = (1.4 \pm 0.5) \cdot 10^{-3} \\ L_5^r(M_\eta) = (2.2 \pm 0.5) \cdot 10^{-3} \end{array} \quad \Rightarrow \quad \frac{f_\eta}{f_\pi} = 1.3 \pm 0.05$$

Comment: SU(3) ChPT has more observables than SU(2) ChPT to extract the LECs

The GL coefficients

List of SU(3) GL coefficients for $\mu = M_\varrho$:

J. Bijnens, G. Ecker and J. Gasser, hep-ph/9411232

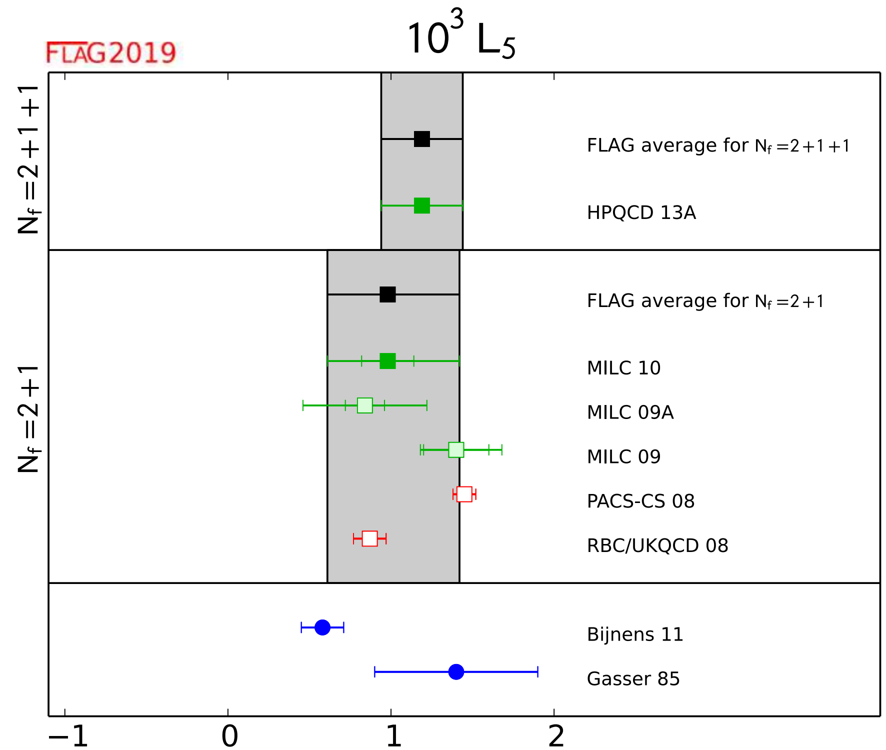
i	$L_i^r(M_\rho) \times 10^3$	source	Γ_i
1	0.4 ± 0.3	$K_{e4}, \pi\pi \to \pi\pi$	$3/32$
2	1.35 ± 0.3	$K_{e4}, \pi\pi \to \pi\pi$	$3/16$
3	-3.5 ± 1.1	$K_{e4}, \pi\pi \to \pi\pi$	0
4	-0.3 ± 0.5	Zweig rule	$1/8$
5	1.4 ± 0.5	$F_K : F_\pi$	$3/8$
6	-0.2 ± 0.3	Zweig rule	$11/144$
7	-0.4 ± 0.2	Gell-Mann–Okubo, L_5, L_8	0
8	0.9 ± 0.3	$M_{K^0} - M_{K^+}, L_5,$ $(2m_s - m_u - m_d) : (m_d - m_u)$	$5/48$
9	6.9 ± 0.7	$\langle r^2 \rangle_V^\pi$	$1/4$
10	-5.5 ± 0.7	$\pi \to e\nu\gamma$	$-1/4$

Other scale:

$$L_k^r(\mu_1) = L_k^r(\mu_2) + \frac{\Gamma_k}{16\pi^2} \ln \frac{\mu_2}{\mu_1}$$

- Extraction from observables is not trivial (e.g. remove emagn. effects → later)
- Some determinations require theoretical input (Zweig rule)
- More recent determinations
  - from phenomenology: J. Bijnens, I. Jemos, arXiv:1103.5945[hep-ph]
  - from lattice simulations: S. Aoki *et. al.* (FLAG review 2019), arXiv:1902.08191

Example: Results for L_5

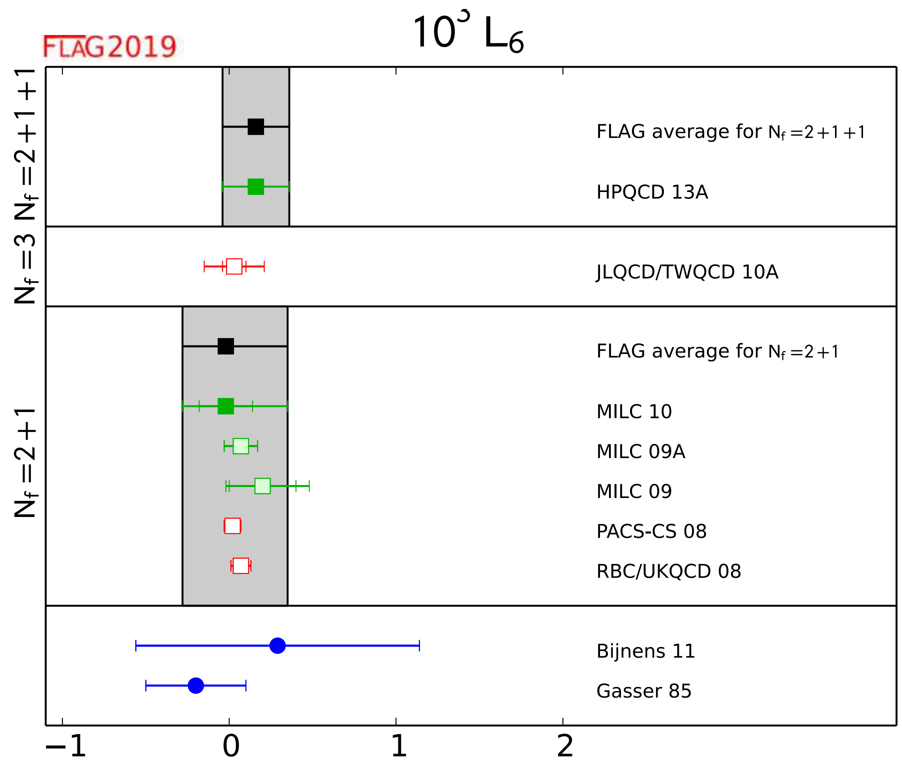


S. Aoki *et. al.* (FLAG review 2019), arXiv:1902.08191

General observation:

Determination of "*symmetry breakers*" (L_4 - L_8) from experiment more difficult than "*symmetry preservers*"

Example: Results for L_6



S. Aoki *et. al.* (FLAG review 2019), arXiv:1902.08191

General observation:

Determination of "*symmetry breakers*" (L_4 - L_8) from experiment more difficult than "*symmetry preservers*"

Systematics of the chiral expansion

Systematics of the derivative expansion

We have seen:

- ChPT is an expansion in (small) external momenta and quark masses
  ➥ the chiral lagrangian contains infinitely many terms

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots$$

- 1-loop contributions with vertices of $\mathcal{L}_2$ contribute like tree-level terms of $\mathcal{L}_4$

How does this generalise to higher loops?

Systematics of the derivative expansion

Consider an amplitude (truncated GF) with

e.g.:

V_d = # vertices with d derivatives

I = # internal lines

L = # loops

$$\mathcal{A} \propto \int d^4 l_1 \ldots d^4 l_L \, \frac{1}{k_1^2 - M^2} \cdots \frac{1}{k_I^2 - M^2} \cdot p^{\sum_d d \cdot V_d}$$

$\mathrm{O}(p^2)$ $\qquad k_i = l_i \pm p_i$

$$\mathcal{A} \propto p^{\left[4L - 2I + \sum_d d \cdot V_d\right]} F(p) \equiv p^{2N+2} F(p)$$

F : dimless function

Use relation $\quad L = I - \sum_d V_d + 1 \quad$ and get rid of I

$$N = L + \sum_d \left(\frac{d-2}{2}\right) V_d$$

Systematics of the derivative expansion

$$\mathcal{A} \propto p^{2N+2} F(p) \qquad N = L + \sum_d \left(\frac{d-2}{2}\right) V_d$$

Note: $N \geq 0$

- Relation between the momentum ($D = 2N+2$) and loop (L) expansion

- The larger $N \geq 0$ the smaller the contribution of this diagram
  Recall: The dimless expansion parameter is $p^2/(4\pi f)^2$

- For a given N only a finite number of
  - loops ($L \leq N$)
  - vertices V_d ($d \leq 2N+2$)

  contribute

$N = 0 \Rightarrow L = 0$ & $d = 2$

$N = 1 \Rightarrow L = 1$ & $d = 2$

$L = 0$ & $d = 4$, $V_4 = 1$

$N = 2 \Rightarrow L = 2$ & $d = 2$

$L = 1$ & $d = 4$, $V_4 = 1$

$L = 0$ & $d = 6$, $V_6 = 1$

➡ The chiral Lagrangian $\mathcal{L}_d$ is needed only up to $d \leq 2N+2$

➡ Only a finite number of LECs enter for a given N
once these are known ChPT can make predictions

Systematics of the derivative expansion

Caveat:
divergent loop integrals

$$F(p) \longrightarrow F\left(\frac{p}{\Lambda}\right) \qquad F\left(\frac{p}{\mu}, \epsilon\right)$$

cut-off DimReg

Possibilities ($k \geq 0$) $F\left(\frac{p}{\Lambda}\right)$:

$$\left(\frac{p}{\Lambda}\right)^{2k}$$ convergent

$$\left(\frac{\Lambda}{p}\right)^{2k}$$ power divergence

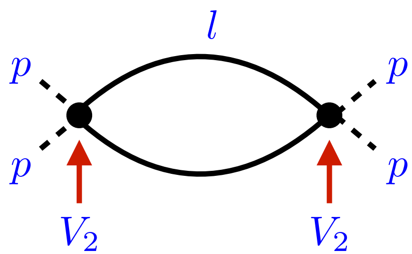
not present in Dim Reg

$$\ln \frac{p^2}{\Lambda^2}$$ logarithmic divergence

cancelled by ren. LECs in $\mathcal{L}_{2N+2}$

Systematics of the derivative expansion

Example: 1-loop ππ scattering



$$(1) \qquad p^4 \int d^4 l \, \frac{1}{(l^2 - M_\pi^2)((l+2p)^2 - M_\pi^2)} \sim \ln \frac{\Lambda}{M_\pi}$$

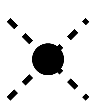
$$(2) \qquad p^2 \int d^4 l \, \frac{l^2}{(l^2 - M_\pi^2)((l+2p)^2 - M_\pi^2)} \sim \Lambda^2$$

$$(3) \qquad \int d^4 l \, \frac{l^2 (l+2p)^2}{(l^2 - M_\pi^2)((l+2p)^2 - M_\pi^2)} \sim \Lambda^4$$

- In DimReg all 3 integrals lead to $B_0(4p^2, M_\pi^2) = \frac{1}{16\pi^2}\left(\frac{2}{\epsilon} - \ln M_\pi^2 + F(4p^2, M_\pi^2)\right)$ finite function

  and $A_0(M_\pi^2)$

- The log divergence $\propto p^4 \cdot \frac{2}{\epsilon}$ is absorbed by

 $\propto p^4 \cdot L \qquad L \longrightarrow L^{re} - \left(\frac{2}{\epsilon} + \ldots\right.$

Systematics of the derivative expansion

$$\mathcal{A} \propto p^{2N+2} F(p) \qquad\qquad N = L + \sum_d \left(\frac{d-2}{2}\right) V_d$$

- ChPT is an expansion in powers of external momenta and quark masses
  At a given *chiral dimension* $D = 2N+2$ only a finite number of
  - loops ($L \leq N$)
  - and vertices of $\mathcal{L}_d$ with $d \leq 2N+2$

  contribute.

	tree	1-loop	2-loop
LO	$\mathcal{L}_2$		
NLO	$\mathcal{L}_4$	$\mathcal{L}_2$	
NNLO	$\mathcal{L}_6$	$\mathcal{L}_4$	$\mathcal{L}_2$

- At a given *chiral dimension* $D = 2N+2$ only a finite number of LECs contribute
  - These are sufficient to renormalise the theory and render the theory finite
    The chiral lagrangian by construction contains all terms compatible with the symmetries
  - Only a finite number of renormalised LECs need to be determined (pheno, lattice) for ChPT to make predictions.

Comment on NNLO

- At NNLO we need $\mathcal{L}_6$ and 2-loop integrals

- The number of terms in the chiral Lagrangian increases rapidly

  $\mathcal{L}_6$ for SU(3) ChPT: 90 + 4 terms

  for SU(2) ChPT: 52 + 4 terms

  probably impossible in practice to determine all the LECs in $\mathcal{L}_6$

- But: Not all LECs contribute to all observables

  ➥ much less need to be determined for a subset of observables like masses and decay constants

- For a review of NNLO results and calculations see

  ChPT beyond 1-loop,
  J. Bijnens, Prog.Part.Nucl.Phys. 58 (2007) 521-586
  hep-ph/0604043

Some loose ends

- Physical external fields
- Electromagnetic isospin breaking
- Anomalies and the Wess-Zumino-Witten term
- Transformation law of the pion fields

Physical vector and axial vector fields

- So far: Axial vector current as an "*interpolating field*" for the pions

$$0 \neq \langle 0|A^a_\mu(0)|\pi^b(\vec{p}\rangle = \delta^{ab} i p_\mu f$$ recall Goldstone theorem

QCD WIs relate axial and vector currents ➠ it is necessary to set up the generating functional with these two currents
➥ obtain QCD correlation functions followed by setting $v = a = 0$

- However: Electromagnetic and weak interactions can be accomodated in ChPT by setting the external fields to their "*physical values*"

➥ This allows to discuss pions interacting with photons and W-bosons

Example: Electromagnetism

Quark charge matrix $$Q = \begin{pmatrix} \frac{2}{3} & & \\ & -\frac{1}{3} & \\ & & -\frac{1}{3} \end{pmatrix} = T^3 + \frac{1}{\sqrt{3}} T^8$$

$$\Rightarrow \qquad v_\mu^3(x) + \frac{1}{\sqrt{3}} v_\mu^8(x) \quad \rightarrow \quad -eA_\mu(x)$$

$$\mathcal{L}_{\rm ext} = v_\mu^a \, \overline{q} \gamma^\mu T^a q \quad \rightarrow \quad -eA_\mu \, \overline{q} \gamma^\mu Q q$$

$$= -eA_\mu \left(\frac{2}{3} \overline{u} \gamma^\mu u - \frac{1}{3} \overline{d} \gamma^\mu d - \frac{1}{3} \overline{s} \gamma^\mu s \right)$$

$$= A_\mu \, j^\mu_{\rm em}$$

Example for application: Compton scattering: $\pi^+(p) + \gamma(q) \rightarrow \pi^+(p') + \gamma(q')$

Comment for SU(2) ChPT:
We need to add a singlett contribution because $Q = \begin{pmatrix} \frac{2}{3} & \\ & -\frac{1}{3} \end{pmatrix} \notin SU(2)$

Example: weak interactions

Weak currents couple to the left-handed quark currents only

$$r_\mu = 0 \qquad l_\mu = -\frac{g}{\sqrt{2}}\left(W_\mu^+ T^+ + h.c.\right) \qquad T^+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{L}_{\text{ext}} = \overline{q}_L l_\mu \gamma^\mu q_L \quad \rightarrow \quad -\frac{g}{\sqrt{2}}\left(V_{ud}\overline{u}_L\gamma^\mu d_L + V_{us}\overline{u}_L\gamma^\mu s_L + h.c.\right)$$

Example for application: semi-leptonic decay $\pi^- \rightarrow \pi^0\, e^-\, \overline{\nu}_e$

Comment: Including the neutral weak current due to Z^0 is more complicated …

Isospin breaking and emagn interaction

$$\mathcal{L}^0_{\mathrm{QCD+QED}} = \overline{q}\big(i\gamma^\mu D_\mu\big)q = \overline{q}\big(i\gamma^\mu D^{\mathrm{color}}_\mu - eA_\mu\gamma^\mu Q\big)q$$

$$= \mathcal{L}^0_{\mathrm{QCD}} - \big(\overline{q}_R Q_R \gamma^\mu q_R + \overline{q}_L Q_L \gamma^\mu q_L\big)A_\mu \qquad Q_R = Q_L = eQ$$

The quark charge matrix breaks flavor (isospin) symmetry

Promote Q_{R} and Q_{L} to spurion fields with transformation behaviour

$$Q_R \xrightarrow{R} RQ_RR^\dagger \qquad Q_R \xrightarrow{P} Q_L$$

$$Q_L \xrightarrow{L} LQ_LL^\dagger$$

Isospin breaking and emagn interaction

Construct the chiral Lagrangian with the additional spurion fields Q_R, Q_L

➥ $\mathrm{tr}[Q_R U Q_L U^\dagger]$ one invariant without derivatives

➥ $$\mathcal{L}_2 = \frac{f^2}{4}\mathrm{tr}[\nabla_\mu U \nabla^\mu U^\dagger] + e^2 C \mathrm{tr}[QUQU^\dagger]$$

- C: LEC not determined by chiral symmetry
- Recall: The photon field is contained in the covariant derivative
- Power counting: $Q \sim p \quad \Rightarrow \quad \mathscr{L}_2$ is the LO Lagrangian
- Additional terms at NLO: $\mathrm{O}(p^2Q^2, m_\mathrm{q}Q^2, Q^4)$

  In total *17 additional terms* to the Gasser-Leutwyler Lagrangian $\mathscr{L}_4$

  with *17 additional LECs* K_i

  R. Urech, hep-ph/9405341

  - provide the neccessary counter terms for 1-loop diagrams.
  - hard to determine all from phenomenology

Isospin breaking and emagn interaction

The $O(Q^2)$ term contributes to the charged pion and kaon masses

$$e^2 C \mathrm{tr}[QUQU^\dagger] = -\frac{2e^2C}{f^2}\left(\pi^+\pi^- + K^+K^-\right) + O(\pi^4)$$

$$\Delta M^2_{\pi^\pm} = \frac{2e^2C}{f^2}$$

$$\Delta M^2_{K^\pm} = \Delta M^2_{\pi^\pm}$$

Exercise:
Derive these results

$$\Delta M^2_{\pi^0} = \Delta M^2_{K^0} = \Delta M^2_{\overline{K}^0} = \Delta M^2_{\eta} = 0$$

- Electromagnetic shift is the same for $\pi^\pm$ and $K^\pm$ *"Dashen's theorem"* (1969)
- Violations of Dashen's theorem from loop diagrams of $O(m_q Q^2)$

Isospin breaking and emagn interaction

$$\left(M_{\pi^\pm}^2 - M_{\pi^0}^2\right)_{\rm exp} = \left(M_{\pi^\pm}^2 - M_{\pi^0}^2\right)_{\rm QCD} + \left(M_{\pi^\pm}^2 - M_{\pi^0}^2\right)_{\rm em}$$

$$\left(M_{\pi^\pm} - M_{\pi^0}\right)_{\rm QCD} = 0.17 \pm 0.03\,\mathrm{MeV}$$

strong isospin breaking
$\sim (m_u - m_d)^2$

$$\Rightarrow \quad \left(M_{\pi^\pm} - M_{\pi^0}\right)_{\rm em} = 4.43 \pm 0.03\,\mathrm{MeV}$$

Gasser, Leutwyler, 1985

Isospin breaking in the pion masses is almost entirely due to emagn effects !

Use Dashen's theorem to estimate the emagn contribution in the kaon masses:

$$\left(K_{\pi^\pm} - M_{K^0}\right)_{\rm em} \approx \left(M_{\pi^\pm} - M_{\pi^0}\right)_{\rm em} \frac{M_\pi}{M_K} \approx 1.25\,\mathrm{MeV}$$

$$\left(K_{\pi^\pm} - M_{K^0}\right)_{\rm exp} \approx -3.934\,\mathrm{MeV}$$

References

Most of what was covered here can be found in:

- Jürg Gasser: ***Light-quark dynamics***, hep-ph/0312367
  ➠ 1-loop pion mass

- Heinrich Leutwyler: ***Principles of chiral perturbation Theory***, hep-ph/9406283
  ➠ Anomalies, Transformation of pion fields

- Maarten Golterman: ***Applications of chiral perturbation theory to lattice QCD***, arXiv:0912.4042 [hep-lat]
  ➠ Systematics of chiral expansion

- Jürg Gasser and Heinrich Leutwylwer: ***Chiral Perturbation Theory: Expansions in the mass of the strange quark***, Nucl. Phys. B250 465
  ➠ Determination L_5

Exercises

For possible exercises see

- slide 11
- slide 35
- see Lect 2, slide 22:
  Convince yourself that the chiral condensate in massless QCD (ChPT) does not receive pion loop corrections