Introduction to Chiral Perturbation Theory

Oliver Bär

Humboldt-Universität zu Berlin

EuroPLEx web-school

Oct 09 2020

Outline

- Lecture I: Introduction to Chiral Perturbation Theory (ChPT) - massless case
- Lecture 2: Introduction to Chiral Perturbation Theory (ChPT) - nonzero quark masses
- Lecture 3: Some ChPT applications in Lattice QCD
- Lecture 4: Beyond pions: Baryon ChPT and Lattice QCD

Goal: Elementary and pedagogical introduction to ChPT
emphasis on the theoretical foundation

Prerequisite: Introductory course(s) on QFT

References

There exist MANY introductions to ChPT on the arXiv / internet, for instance

- Heinrich Leutwyler: ***Chiral Dynamics***, hep-ph/0008124
- Bastian Kubis, ***An introduction to chiral perturbation theory***, hep-ph/0703274
- Gilberto Colangelo and Gino Isidori, ***An introduction to ChPT***, hep-ph/0101264
- Stefan Scherer, ***Chiral perturbation theory: An introduction and recent results in the one-nucleon sector***, arXiv:0908.3425[hep-ph]
- Gerhard Ecker, ***Strong interactions of light flavours***, hep-ph/0011026

Monograph:

- Stefan Scherer and Matthias Schindler, ***A primer for chiral perturbation theory***
  Lect. Notes Phys. 830 (2012) pp.1-338 (Springer)

Classic papers

- S. Weinberg, Physica A 96 (1979) 327
- J. Gasser and H. Leutwyler, Annals Phys. 158 (1984) 142
- J. Gasser and H. Leutwyler, Nucl. Phys. B250 (1985) 465

Outline Lecture 1

- Massless QCD and its symmetries
- Spontaneous symmetry breaking and the Goldstone theorem
- Chiral Ward identites
- Constructing an effective low-energy theory for massless QCD: ChPT

What is ChPT?

Chiral perturbation theory (ChPT, χPT): An effective field theory for QCD

Based on

- the symmetries of QCD
- an expansion of the Greens functions in powers of
  - small momenta $p << \Lambda_{\text{QCD}}$
  - small quark masses $m_u, m_d, m_s << \Lambda_{\text{QCD}}$

    "expansion around massless QCD"

Example: Pion scattering $\pi^+(p_1)\ \pi^+(p_2) \longrightarrow \pi^+(p_3)\ \pi^+(p_4)$

➥ scattering amplitude
$$T = \frac{2M_\pi^2 - s}{f_\pi^2} + \mathrm{O}(p^4)$$

$s = (p_1 + p_2)^2$

$f_\pi = 92.4\,\mathrm{MeV}$

pion decay constant

expect good description for small p^2

Why effective field theories?

Effective field theories (EFTs) provide a good description of an underlying theory in a restricted energy momentum range

Effective theories are useful if

- calculations in the underlying theory are complicated or even impossible
  e.g.: QCD and pion scattering

- the underlying theory is not known
  - ➥ the effective theory parameterizes systematically our ignorance about the underlying theory

Massless QCD

QCD with light quarks only

Lagrangian:

$$\mathcal{L}_{\rm QCD} = \mathcal{L}^0_{\rm QCD} - \mathcal{L}^m_{\rm QCD}$$

$$\mathcal{L}^0_{\rm QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} + \bar{q}_R i \not{D} q_R + \bar{q}_L i \not{D} q_L$$

$$\mathcal{L}^m_{\rm QCD} = \bar{q}_R M q_L + \bar{q}_L M^\dagger q_R \qquad M = {\rm diag}({\rm m_u}, {\rm m_d}, {\rm m_s})$$

with quark and antiquark triplets $q = (u, d, s)^{\rm T}$ $\bar{q} = (\bar{u}, \bar{d}, \bar{s})$

decomposed in chiral fields $q = \frac{1}{2}(1+\gamma_5)q + \frac{1}{2}(1-\gamma_5)q \equiv q_R + q_L$

For $m_{u,d,s} \ll \Lambda_{\rm QCD} \sim 1{\rm GeV}$ we may naively expect

$$\mathcal{L}_{\rm QCD} = \mathcal{L}^0_{\rm QCD} \text{ + small perturbation}$$

First approximation: drop the mass term!

Less justified for m_s. Later more…

Symmetries of the massless Lagrangian

$$\mathcal{L}^0_{\mathrm{QCD}} = -\frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} + \overline{q}_R i \not{D} q_R + \overline{q}_L i \not{D} q_L$$

is invariant under under *global chiral flavour transformations*

Physics: Gluon interactions preserve quark helicity

$$(q_R, q_L) \longrightarrow (R q_R, L q_L) \qquad R \in U(3)_R$$

$$(\overline{q}_R, \overline{q}_L) \longrightarrow (q_R^{\mathrm{T}} R^\dagger, q_L^{\mathrm{T}} L^\dagger) \qquad L \in U(3)_L$$

Global symmetry group of massless QCD Lagrangian

$$G = SU(3)_R \times SU(3)_L \times U(1)_V \times U(1)_A$$

Vector transformation $V: \; R = L$

Axial vector transformation $A: \; R = L^{-1}$

Associated conserved charges

$$G = SU(3)_R \times SU(3)_L \times U(1)_V \times U(1)_A$$

Symmetry group implies 18 Noether currents

- $U(1)_V$ $\quad V_\mu = \bar{q}\gamma_\mu q = \bar{q}_R\gamma_\mu q_R + \bar{q}_L\gamma_\mu q_L$ $\quad Q_V = \int d^3x V_0$

  Baryon number

- $U(1)_A$ $\quad A_\mu = \bar{q}\gamma_\mu\gamma_5 q = \bar{q}_R\gamma_\mu q_R - \bar{q}_L\gamma_\mu q_L$ $\quad$ broken by axial anomaly

- $SU(3)_R \times SU(3)_L$ $\quad V^a_\mu = \bar{q}\gamma_\mu T^a q$ $\quad$ vector current $\quad Q^a_V$

  $A^a_\mu = \bar{q}\gamma_\mu\gamma_5 T^a q$ $\quad$ axial current $\quad Q^a_A$

SU(3) generators $\quad T^a = \dfrac{\lambda^a}{2}$ ← Gell-Mann matrices

Quantization

Familiar quantization procedures

- canonical: $\mathcal{L}$ $\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}$ $[\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y})$

  $$\mathcal{H} = \pi \partial_0 \phi - \mathcal{L} \qquad H = \int d^3x \mathcal{H}(x)$$

  Difficult for non-abelian gauge theories

- Path integrals

  Path integral representation for Greens functions

  e.g.

  $$G(x,y) = \langle 0|T\phi(x)\phi(y)|0\rangle = \frac{1}{Z}\int \mathcal{D}[\phi]\,\phi(x)\phi(y)\,e^{iS[\phi]}$$

  Hamiltonian implicit, Generator of time translations

  (Lattice: Logarithm of the transfer matrix)

Comment: Need for regulator $\rightarrow$ later

Symmetries of the ground state

Conserved charges commute with the QCD Hamiltonian $H_0 = H_{\mathrm{QCD,massless}}$

$$[H_0, Q_V^a] = [H_0, Q_A^a] = 0$$

➥ Degeneracy of the spectrum : Take eigenstate $|\psi\rangle$ of H_0

✏️ $H_0|\psi\rangle = E|\psi\rangle \quad \Rightarrow \quad H_0\, Q_{V,A}^a|\psi\rangle = Q_{V,A}^a\, H_0|\psi\rangle = E\, Q_{V,A}^a|\psi\rangle$

➥ The states $|\psi\rangle$ $Q_V^a|\psi\rangle$ $Q_A^a|\psi\rangle$ have the same energy

Note: Choose $|\psi\rangle$ to be a parity eigenstate

➥ $|\psi\rangle$ $Q_V^a|\psi\rangle$ same parity

$|\psi\rangle$ $Q_A^a|\psi\rangle$ opposite parity

$\Rightarrow$ "Parity doubling" is expected

not observed in Nature !

Symmetries of the ground state

But: Reasoning about degeneracy requires invariance of QCD vacuum $|0\rangle$

$Q_V^a|0\rangle = 0$ $Q_A^a|0\rangle = 0$ *Wigner-Weyl-mode*

Alternative: $Q_V^a|0\rangle = 0$ $Q_A^a|0\rangle \neq 0$ *Nambu-Goldstone-mode*
degenerate ground state,
spontaneous symmetry breaking

➥
- Existence of 8 massless Nambu-Goldstone bosons (NGBs)
  Goldstone theorem
- The axial charges Q_A^a create NGBs

$$Q_A^a|\psi\rangle = |\psi, NGB_1, \ldots, NGB_M, \ldots\rangle$$

≠ 1 particle state, no parity doubling in spectrum

➥ Nambu-Goldstone mode is realised in QCD ! "Explains"

- Absence of parity doubling in spectrum
- Pions and Kaons as pseudo NGBs
  Not massless due to nonzero but small quark masses

Comment: Vector symmetry is not spontaneously broken (*Vafa-Witten theorem*)

The Goldstone theorem

Goldstone theorem

Consider a quantum field theory with the following properties

a) There exists a conserved current $\partial^\mu J_\mu = 0 \qquad Q = \int d^3x J_0(x^0, \vec{x})$

b) There exists an operator O such that $\langle 0|[Q, O]|0\rangle \neq 0$

Note: b) implies $Q|0\rangle \neq 0$

➥ The Goldstone theorem applies:

There exists a massless particle $|NGB(\vec{p})\rangle \qquad p^2 = 0$

and $\langle 0|J_\mu|NGB(\vec{p})\rangle\langle NGB(\vec{p})|O|0\rangle \neq 0$

$$\Rightarrow \quad 0 \neq \langle 0|J_\mu|NGB(\vec{p})\rangle = ip_\mu f$$

↑ Lorentz covariance

The current couples to the NGBs
"interpolating field"

It determines their quantum numbers

Sketch of proof

$$0 \neq \chi = \int d^3\vec{x}\ \langle 0|J^0(t,\vec{x})\phi(0)\rangle - \langle 0|\phi(0)J^0(t,\vec{x})|0\rangle$$

Use

- Completeness of states $1 = |0\rangle\langle 0| + \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}|\vec{p},\lambda\rangle\langle\vec{p},\lambda|$
- Translation invariance $J^\mu(x) = e^{iP_\mu x^\mu} J^\mu(0) e^{-iP_\mu x^\mu}$

➥ $$0 \neq \chi = \int d^3\vec{x} \sum_\lambda \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{e^{-ip_\mu x^\mu}}{2E_p}\langle 0|J^0(0)|\vec{p},\lambda\rangle\ \langle\vec{p},\lambda|\phi(0)\rangle - \frac{e^{+ip_\mu x^\mu}}{2E_p}\ldots$$

sum over $\vec{p}=0$ states only

$$= \sum_\lambda \int d^3\vec{p}\ \delta^3(\vec{p})\Big(\frac{e^{-iE_{p,\lambda}t}}{2E_p}\langle 0|J^0(0)|\vec{p},\lambda\rangle \ldots \qquad = {\sum_\lambda}' c_\lambda e^{-iE_\lambda t} - \ldots$$

$\chi = \text{const}$ $\Rightarrow$ contribution of states with $E_\lambda > 0$ must vanish

$\chi \neq 0$ $\Rightarrow$ at least one state with $\vec{p} = E_p = 0$ must exist and contribute, *NG boson*

Comments

- For vacuum state not to contribute we need $\langle 0|O|0\rangle = 0$
- Simple but sloppy argument: NGB has a four vector $p^\mu = (E_{\vec{p}} = 0, \vec{p} = 0)$
  - ➥ massless particle at rest ✘

Sketch of proof (2)

$$\langle 0|[J^\mu(x),\phi(y)]|0\rangle = \frac{\partial}{\partial x_\mu}\int d\mu^2 \rho(\mu^2)\Big(D(x-y,\mu^2) - D(y-x,\mu^2)\Big)$$

$$\rho(\mu^2) = \sum_\lambda \delta(\mu^2 - m_\lambda^2) c_\lambda$$ *Källén-Lehmann spectral density*

To derive this use: complete set of states and translation invariance (as before)

$$\sum_\lambda \int \underbrace{\frac{d^3\vec{p}}{(2\pi)^3}\,\frac{1}{2E_p}}_{\equiv D(x-y,m_\lambda^2)} e^{-ip(x-y)}\langle 0|J^\mu(0)|\vec{p},\lambda)\rangle\langle\vec{p},\lambda)|\phi(0)|0\rangle - \frac{1}{2E_p}e^{-ip(y-x)}\ldots$$

and

- Lorentz covariance: $\langle 0|J^\mu(0)|\vec{p},\lambda)\rangle = ip^\mu f_\lambda(p^2)$
- Get ip^μ by $\frac{\partial}{\partial x_\mu} D(x-y,m_\lambda^2)$
- Introduce integral over the spectral density

details e.g. in S. Weinberg, *The Quantum Theory of Fields II*

Sketch of proof (2)

$$\langle 0|[J^\mu(x),\phi(y)]|0\rangle = \frac{\partial}{\partial x_\mu}\int d\mu^2 \rho(\mu^2)\Big(D(x-y,\mu^2) - D(y-x,\mu^2)\Big)$$

$$\rho(\mu^2) = \sum_\lambda \delta(\mu^2 - m_\lambda^2) c_\lambda$$ *Källén-Lehmann spectral density*

Use our two assumptions:

1) Current conservation $\partial_\mu J^\mu = 0$

$$(\Box + \mu^2) D(x-y,\mu^2) = 0$$

$$\Rightarrow \qquad 0 = \int d\mu^2 \rho(\mu^2)\,\Box\Big(D(x-y,\mu^2) - D(y-x,\mu^2)\Big)$$

$$\rho(\mu^2)\mu^2 \stackrel{!}{=} 0$$

$$\rho(\mu^2) = k\,\delta(\mu^2)$$ k: constant

If $k \neq 0$ then there exists a massless state, the Nambu-Goldstone boson

Sketch of proof (2)

$$\langle 0|[J^\mu(x),\phi(y)]|0\rangle = \frac{\partial}{\partial x_\mu}\int d\mu^2 \rho(\mu^2)\Big(D(x-y,\mu^2) - D(y-x,\mu^2)\Big)$$

$$\rho(\mu^2) = k\,\delta(\mu^2)$$ k: constant

Use our two assumptions:

2) Consider the matrix element for $\mu = 0$ and $x_0 = y_0$

and use $$\frac{\partial}{\partial x_0}\Big(D(x-y,0) - D(y-x,0)\Big) = -i\delta^{(3)}(\vec{x}-\vec{y})$$

$\Rightarrow$ $$\langle 0|[J^0(x),\phi(y)]|0\rangle = -ik\delta^{(3)}(\vec{x}-\vec{y})$$

➥ $$0 \neq \langle 0|[Q,\phi(0)]|0\rangle = -ik$$

$k \neq 0$ $\Rightarrow$ there exists a massless state, the Nambu-Goldstone boson !

Comments

- No reference to a "mexican hat potential" in the Lagrangian
  - ➥ applies also to theories without scalar fields in the Lagrangian

- No reference to a potential at all
  - ➥ applies also to a free massless theory

Goldstone theorem - a trivial example

Free scalar field theory $\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi$

Invariant under $\phi(x) \longrightarrow \phi'(x) = \phi(x) + \lambda$

a) There exists a conserved current $J^\mu(x) = \partial^\mu \phi(x)$

b) There exists another operator $O(0) = \phi(0)$

with $$\int d^3x \, \langle 0 | [J^0(0, \vec{x}), \phi(0, \vec{0})] | 0 \rangle = -i \neq 0$$

Goldstone theorem applies $\Rightarrow$ there exists a massless particle

and $\langle 0 | J^\mu(0) | \vec{p} \rangle = -ip^\mu \neq 0$

Goldstone theorem - a less trivial example

The linear sigma model:
four real fields, $a = 0, \ldots, 3$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - \frac{g}{4}(\phi^a\phi^a - v^2)^2$$

Invariant under $\phi^a \longrightarrow \phi'^a = R^{ab}\phi^b \qquad R \in O(4)$

For $v^2 > 0$ we have spontaneous symmetry breaking: $O(4) \longrightarrow O(3)$

Exercise: Work out the details !

See Jürg Gasser, hep-ph/0312367

Goldstone theorem for massless QCD

Applied to massless QCD :

a) Conserved currents $J_\mu = A^a_\mu$

b) Operator $O = P^b = \overline{q}\gamma_5 T^b q$ Pseudo scalar density $\langle 0|O|0\rangle = 0$

same symmetry properties as axial current

$$\Rightarrow \quad \langle 0|[Q^a_A, P^b]|0\rangle = -\delta^{ab}\frac{1}{3}\langle 0|\overline{q}q|0\rangle$$

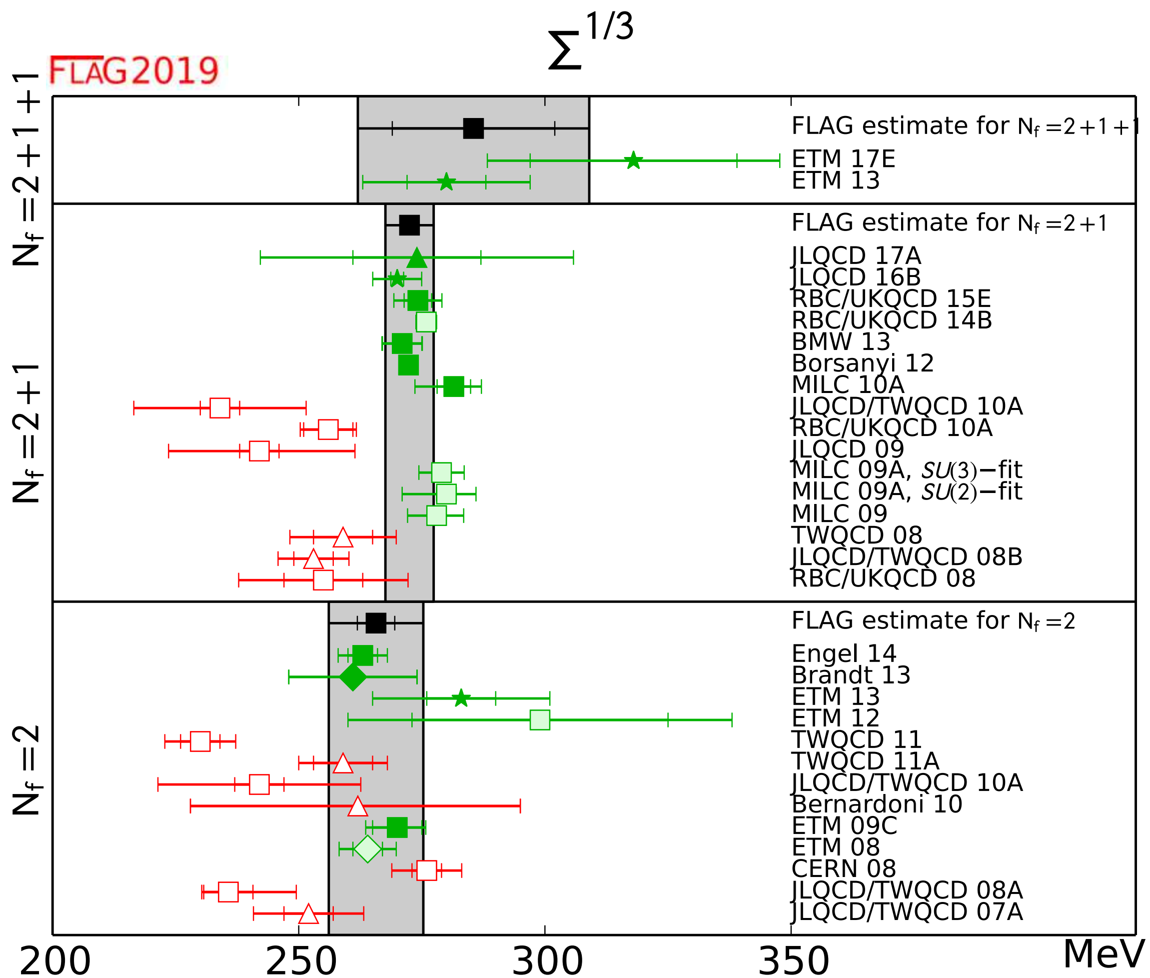
$$= \delta^{ab}\Sigma$$

Chiral condensate

order parameter for

spontaneous chiral symmetry breaking

Lattice QCD results for the chiral condensate



From: Flag review 2019
arXiv: 190208191 [hep-lat]

Note: The chiral condensate is not accessible in perturbation theory !
➥ non-perturbative calculations are necessary

Goldstone theorem for massless QCD

Applied to massless QCD :

a) Conserved currents $J_\mu = A^a_\mu$

b) Operator $O = P^b = \bar{q}\gamma_5 T^b q$ Pseudo scalar density

$$\Rightarrow \quad \langle 0|[Q^a_A, P^b]|0\rangle = -\delta^{ab}\frac{1}{3}\langle 0|\bar{q}q|0\rangle$$
$$= \delta^{ab}\Sigma$$

Chiral condensate
order parameter for
spontaneous chiral symmetry breaking

Lattice QCD results show that the chiral condensate is nonzero !

$$\Rightarrow \quad |NGB(\vec{p})\rangle = |\pi^a(\vec{p})\rangle$$

pions, kaons and eta (pseudo scalars!)

$$0 \neq \langle 0|A^a_\mu|\pi^b(\vec{p})\rangle = \delta^{ab} i p_\mu f \leftarrow$$

pion decay constant
in massless QCD

Note: Non-vanishing chiral condensate is sufficient but not necessary condition for symmetry breaking

NGB scattering

Scattering of NGBs? Complicated to compute in QCD

Sketch: Consider the matrix element

(drop flavor indices for simplicity)

Argument by J. Gasser 2008

$$G_\mu(p_3, p_4; p_1) \equiv \ {}_{\text{out}}\langle \pi(p_3)\pi(p_4)|A_\mu(0)|\pi(p_1)\rangle_{\text{in}}$$

according to LSZ formula $= \dfrac{f q_\mu}{q^2} T(p_3, p_4; p_1) + R_\mu$ T: scattering amplitude

singular regular for $q_\mu \longrightarrow 0$

$q_\mu = (p_3 + p_4 - p_1)_\mu$

current conservation $0 = q^\mu G_\mu = fT + q^\mu R_\mu \longrightarrow fT$

$\Rightarrow$ NGBs are *non-interacting at vanishing momenta* !

Summary so far

- Massless QCD exhibits *spontaneous chiral symmetry breaking* (SSB)

$$SU(3)_R \times SU(3)_L \xrightarrow{\text{SSB}} SU(3)_V$$

- Consequences
  - Existence of 8 weakly interacting NGB
  - Mass gap between NGBs and rest of spectrum with $M_{\text{had}} \gtrsim \Lambda_{\text{QCD}} \approx 1\,\text{GeV}$

- Idea: Construct a low-energy effective theory for QCD = ChPT
  Weinberg, Gasser, Leutwyler, …
  - Contains only the NGBs as active degrees of freedom
  - Organised in powers of NGB momenta $\frac{p^2}{\Lambda_{\text{QCD}}^2} \gg \frac{p^4}{\Lambda_{\text{QCD}}^4} \gg \dots$
  - Reproduces the QCD symmetries (Ward identities)

Chiral Ward identities

Chiral Ward identities

Functional integrals for Greens functions: $$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[F]\, O[F]\, e^{iS^0_{\rm QCD}[F]}$$

composite operator

quark and gluon fields

Consider an infinitesimal <u>*local*</u> axial transformation:

$$q(x) \to q'(x) = q(x) + \delta q(x) \qquad \delta q(x) = i\omega^a(x) T^a \gamma_5 q(x)$$

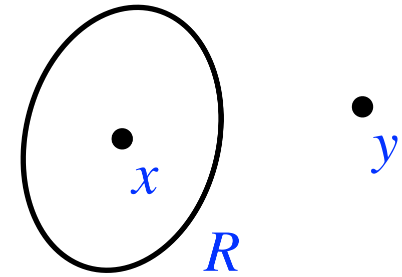
$$\overline{q}(x) \to \overline{q}'(x) = \overline{q}(x) + \delta\overline{q}(x) \qquad \delta\overline{q}(x) = i\overline{q}(x) T^a \gamma_5 \omega^a(x)$$

Since $\mathcal{D}[F] = \mathcal{D}[F']$ ($\mathrm{Tr}\, T^a = 0$!) we find

$$\begin{aligned} \int \mathcal{D}[F'] O[F'] e^{iS^0_{\rm QCD}[F']} &= \int \mathcal{D}[F] O[F'(F)] e^{iS^0_{\rm QCD}[F'(F)]} \\ &= \int \mathcal{D}[F] (O[F] + \delta O[F]) e^{iS^0_{\rm QCD}[F] + i\delta S^0[F]} \\ &= \int \mathcal{D}[F] (O[F] + \delta O[F]) (1 + i\delta S^0[F]) e^{iS^0_{\rm QCD}[F]} \end{aligned}$$

➥ $\langle \delta O + i\delta S^0\, O \rangle = 0$ *Masterformula for WIs*

Chiral Ward identities



Consider axial trafo with $\omega^a(x)=0$ outside region R

$$\delta S^0 = -\int_R d^4x\, \partial_\mu \omega^a(x) A^{\mu,a}(x) = \int_R d^4x\, \omega^a(x) \partial_\mu A^{\mu,a}(x)$$

Use e.g. $O= A^{\nu,b}(y)$ with y outside of R $\Rightarrow$ $\delta O=0$

$\langle \delta O + i\delta S^0\, O \rangle = 0$ turns into

$$\langle T \partial_\mu A^{\mu,a}(x) A^{\nu,b}(y) \rangle = 0$$

conservation of the axial vector current

Chiral Ward identities

More examples:

1. Different *O(y)* $\langle T\partial_\mu A^{\mu,a}(x)P^b(y)\rangle = 0$ $P^a(x) = \overline{q}(x)\gamma_5 T^a q(x)$

2. Vector transformations $\langle T\partial_\mu V^{\mu,a}(x)O^b(y)\rangle = 0$

3. Quark masses non-zero $\langle T\partial_\mu A^{\mu,1}(x)O^b(y)\rangle = \ i(m_u + m_d)\langle T\partial_\mu P^1(x)O^b(y)\rangle$

PCAC relation

Note: The derivation requires *local* chiral transformations!

Exercise: Derive 2 and 3

See Martin Lüscher, hep-lat/9802029

Chiral Ward identities

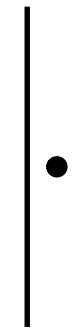
More WIs: $O(y)$ with y inside of R $\Rightarrow$ $\delta O \neq 0$ *Integrated WIs*

Example: $O(y) = P^b(y)$

➥ $$\int_R d^4x\, \omega^a(x) \langle \partial_\mu A^{\mu,a}(x) P^b(y) \rangle = -\delta^{ab} \omega^b(y) \cdot \frac{\langle \bar{q}q(y) \rangle}{3}$$

Use

- $\omega^a(x) \to 1$ in R
- Stokes theorem $\int_R d^4x\, \langle \partial_\mu A^{\mu,a} \ldots = \int_{\partial R} dn_\mu(x)\, \langle A^{\mu,a} \ldots$
- region R: volume between two time slices “sandwiching” the origin $y = 0$



R

➥ $$\int d^3x\, \langle A^{0,a}(+\epsilon, \vec{x}) P^b(0) - P^b(0) A^{0,a}(-\epsilon, \vec{x}) \rangle = -\frac{\delta^{ab}}{3} \langle \bar{q}q(0) \rangle$$

in short: $$\langle [Q_A^a, P^b] \rangle = -\frac{\delta^{ab}}{3} \langle \bar{q}q \rangle$$

cp. with slide 24 !

Chiral Ward identities

More examples

$$[Q_A^a, A_0^b(x)] = i f^{abc} V_0^c(x)$$

f^{abc}: su(3) structure constants

$$[Q_A^a, V_0^b(x)] = i f^{abc} A_0^c(x)$$

$$[Q_V^a, V_0^b(x)] = i f^{abc} V_0^c(x)$$

$$[Q_V^a, A_0^b(x)] = i f^{abc} A_0^c(x)$$

so called
Current algebra

hold in Greens functions with arbitrary external operators with support outside R

e.g.: $\langle 0|T[Q_V^a, V_0^b(x)]O_{\rm ext}(y,z,\ldots)|0\rangle = \langle 0|T i f^{abc} V_0^c(x) O_{\rm ext}(y,z,\ldots)|0\rangle$

Chiral Ward identities

Things to keep in mind:

- The derivation of the chiral Ward identities requires *local chiral transformations*
- The chiral Ward identities relate correlation functions with different currents and densities

Pion pole dominance

recall slide 18

$$\langle 0|T A^a_\mu(x) A^b_\nu(y)|0\rangle = \int d\mu^2 \rho^{ab}(\mu^2)\partial_\mu\partial_\nu D(x-y,\mu^2)$$ for $x_0 > y_0$

➥ Fourier transform $$= \int d\mu^2 \rho^{ab}(\mu^2) p_\mu p_\nu \frac{i}{p^2-\mu^2+i\epsilon}$$

$$= i\delta^{ab} f^2 \frac{p_\mu p_\nu}{p^2+i\epsilon} \; + \;$$ less singular terms

For small pion momenta the pion contribution dominates, *pion pole dominance*

Note: this is an assumption about the absence of light (or massless) particles other than the pions

Constructing an effective theory

Chiral perturbation theory

Construction of an effective pion theory

- Goal: Construct an effective theory for QCD that
  - only contains the pions as active degrees of freedoms
  - reproduces the pion contribution in correlation fuinctions
    which is expected to be the dominant contribution at small momenta
  - reproduces the chiral WIs of QCD

Pedestrian way to construct ChPT

Let us construct a field theory for the pion fields:

$$\mathcal{L}_\pi = \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a + b_0 \pi^a \pi^a$$ quadratic in pion fields

$$+ b_1 (\pi^a \pi^a)^2 + \ldots$$ quartic, no derivative

$$+ c_1 (\partial_\mu \pi^a \pi^a)^2 + c_2 (\partial_\mu \pi^a \partial^\mu \pi^a)(\pi^b \pi^b) + \ldots$$ quartic, two derivatives

$$+ \ldots$$ quartic, four derivatives, …

$$V_\mu^a = v_1 f^{abc} \partial_\mu \pi^b \pi^c + \ldots$$

$$A_\mu^a = a_1 \partial_\mu \pi^a + a_2 \partial_\mu \pi^a (\pi^b \pi^b) + a_3 \pi^a (\partial_\mu \pi^b \pi^b) + \ldots$$

b_k, c_k, v_k, a_k: coupling constants, a priori unknown (“low-energy constants” or LECs)

Comments:

- So far: General expressions with proper symmetries (scalar, vector, axial vector)
- Order principle: number of derivatives, *“Derivative expansion”*

Pedestrian way to construct ChPT

Let us construct a field theory for the pion fields:

$$\mathcal{L}_\pi = \frac{1}{2}\partial_\mu\pi^a\partial^\mu\pi^a + b_0\pi^a\pi^a$$ quadratic in pion fields

$$+ b_1(\pi^a\pi^a)^2 + \ldots$$ quartic, no derivative

$$+ c_1(\partial_\mu\pi^a\pi^a)^2 + c_2(\partial_\mu\pi^a\partial^\mu\pi^a)(\pi^b\pi^b) + \ldots$$ quartic, two derivatives

$$+ \ldots$$ quartic, four derivatives, …

$$V_\mu^a = v_1 f^{abc}\partial_\mu\pi^b\pi^c + \ldots$$

$$A_\mu^a = a_1\partial_\mu\pi^a + a_2\partial_\mu\pi^a(\pi^b\pi^b) + a_3\pi^a(\partial_\mu\pi^b\pi^b) + \ldots$$

Constraints:

1. Massless pions $\Rightarrow$ $b_0 = 0$
2. Non-interacting for vanishing momenta $\Rightarrow$ $b_k = 0$ for $k = 1, \ldots$

➥ no vertex without at least two derivatives

Pedestrian way to construct ChPT

$$\mathcal{L}_\pi = \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a + c_1(\partial_\mu \pi^a \pi^a)^2 + c_2(\partial_\mu \pi^a \partial^\mu \pi^a)(\pi^a \pi^a) + \ldots$$

$$V_\mu^a = v_1 f^{abc} \partial_\mu \pi^b \pi^c + \ldots$$

$$A_\mu^a = a_1 \partial_\mu \pi^a + a_2 \partial_\mu \pi^a (\pi^b \pi^b) + a_3 \pi^a (\partial_\mu \pi^b \pi^b) + \ldots$$

Constraints:

3. We want the effective theory to reproduce the chiral WIs of QCD

➥ The coefficients must satisfy: $c_1 = -c_2$ $\quad a_2 = -a_3$

$6c_1 a_1^2 = -1$ $\quad v_1 = i$ etc.

Only one undetermined coefficient

How to see this (in principle):

1. Compute suitable Greens functions in the effective theory
2. Impose the chiral Ward identities

⇒ Can only be satisfied for the given coefficients

Pedestrian way to construct ChPT

$$\mathcal{L}_\pi = \frac{1}{2}\partial_\mu\pi^a\partial^\mu\pi^a + c_1(\partial_\mu\pi^a\pi^a)^2 + c_2(\partial_\mu\pi^a\partial^\mu\pi^a)(\pi^a\pi^a) + \dots$$

$$V_\mu^a = v_1 f^{abc}\partial_\mu\pi^b\pi^c + \dots$$

$$A_\mu^a = a_1\partial_\mu\pi^a + a_2\partial_\mu\pi^a(\pi^b\pi^b) + a_3\pi^a(\partial_\mu\pi^b\pi^b) + \dots$$

Fix the undetermined coefficient:

Compute the matrix element $\langle 0|A_\mu^a(x)|\pi^b(\vec{p}\rangle \equiv \delta^{ab} i p_\mu f e^{-i\vec{p}\vec{x}}$

➥ $a_1 = -f$ pion decay constant in the chiral limit

Express remaining coefficients in terms of f

➥ $c_1 = -c_2 = \frac{1}{6f^2}$ $a_3 = -a_2 = \frac{2}{3f^2}$

Pedestrian way to construct ChPT

Lessons to be learned:

- Construction principle for the chiral Lagrangian:
  - Write down the most general Lagrangian for the pion fields, organised by powers of derivatives and associated with unknown couplings (LECs)
  - Impose the chiral Ward identities for Greens functions and find relations for the couplings
- But: Tedious procedure
- More efficient: Construct and match the generating functional for Greens functions. All chiral Ward identities are obtained from this single expression !

Standard way to construct ChPT

Effective pion theory should reflect the chiral SSB as QCD, i.e.

1. Lagrangian invariant under $G = SU(3)_R \times SU(3)_L$
2. Vacuum invariant only under $H = SU(3)_{R=L}$

Question: How do the pion fields transform under G ?

Answer: There exists the 1-to-1 map

$$\pi^a(x) \longrightarrow U(x) = \exp\left[\frac{2i}{\tilde{f}}\pi^a(x)T^a\right] \qquad \dim[\tilde{f}] = 1$$

that maps the pion fields onto the *coset space* $G/H \simeq SU(3)$

Why $U(x)$? Transforms extremely simple under $R, L \in G$

$$U(x) \xrightarrow{G} R\,U(x)\,L^\dagger$$

later more if time permits …

Standard way to construct ChPT

Construct invariant Lagrangian using $U \longrightarrow RUL^\dagger$ $\quad U^\dagger \longrightarrow LU^\dagger R^\dagger$

We need both !

Invariants

- No derivatives: $\mathrm{Tr}[UU^\dagger] \longrightarrow \mathrm{Tr}[RUL^\dagger LU^\dagger R^\dagger] = \mathrm{Tr}[UU^\dagger]$ invariant, but constant ➥ drop

- One derivative: $\mathrm{Tr}[\partial_\mu UU^\dagger]$ not Lorentz invariant

- Two derivatives: $\mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger]$ $\quad \mathrm{Tr}[\partial_\mu UU^\dagger \partial^\mu UU^\dagger] = -\mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger]$

  not independent because of $\quad 0 = \partial_\mu \left(U^\dagger U\right) = \partial_\mu U^\dagger U + U^\dagger \partial_\mu U$

➥ only one invariant up to two derivatives

Standard way to construct ChPT

With two derivatives only one unique term compatible with all symmetries:
Chiral, Lorentz, Parity, Charge Conj.

$$\mathcal{L}_2 = \frac{\tilde{f}^2}{4}\mathrm{Tr}[\partial_\mu U \partial^\mu U^\dagger]$$

$\tilde{f}$: LEC

Expand U in powers of pion fields

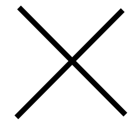
$$U(x) = \exp\left[\frac{2i}{\tilde{f}}\pi^a(x)T^a\right] = 1 + \frac{2i}{\tilde{f}}\pi^a(x)T^a - \frac{2}{\tilde{f}^2}\Big(\pi^a(x)T^a\Big)^2 + \ldots$$

➥ $$\mathcal{L}_2 = \frac{1}{2}\partial_\mu\pi^a\partial^\mu\pi^a + \frac{1}{6\tilde{f}^2}(\partial_\mu\pi^a\pi^a)^2 - \frac{1}{6\tilde{f}^2}(\partial_\mu\pi^a\partial^\mu\pi^a)(\pi^b\pi^b) + \ 6\pi,\ 8\pi,\ \ldots$$

Note: Identification $\tilde{f} = f$ via the axial vector matrix element

First application: $\pi\pi$ scattering

$$\mathcal{L}_{\text{int},4\pi} = \frac{1}{6f^2}\left[(\partial_\mu \pi^a \pi^a)(\partial^\mu \pi^b \pi^b) - (\partial_\mu \pi^a \partial^\mu \pi^a)(\pi^b \pi^b)\right]$$

Implies pion scattering proportional to $\dfrac{p^2}{6f^2}$ 

Note: Non renormalizable ➞ later more

Consider scattering $\pi^+(p_1)\ \pi^+(p_2) \longrightarrow \pi^+(p_1')\ \pi^+(p_2')$

➥ scat amplitude $T = -\dfrac{s}{f^2}$ $\quad s = (p_1 + p_2)^2$

vanishes for $p_1, p_2 \longrightarrow 0$

As expected: Pions (as NGBs) are weakly interacting at small momenta !

References

Most of what was covered here can be found in:

- Jürg Gasser: ***Light-quark dynamics***, hep-ph/0312367
- Heinrich Leutwyler: ***Principles of chiral perturbation Theory***, hep-ph/9406283
- Martin Lüscher: ***Advanced Lattice QCD***, hep-lat/9802029
  ➠ Chiral Ward identities
- Stefan Scherer, Matthias Schindler, ***A Primer for Chiral Perturbation Theory***, Lecture Notes in Physics 830, Springer
  successor of
  Stefan Scherer, Matthias Schindler, ***A Chiral Perturbation Theory Primer,*** hep-ph/0505265

Exercises

For possible exercises see

- slide 23
- slide 32