Integral representation of the modified Bessel function:
(slide 24)

$$K_1(z) = \frac{z}{4}\int_0^\infty dt\ \frac{1}{t^2}\ e^{-\left(t+\frac{z^2}{4t}\right)}$$

Show

$$K_1(z) = \sqrt{\frac{\pi}{2z}}\ e^{-z} \qquad \text{for} \quad z \to \infty$$

Solution: Perform a saddle point expansion for the integral

$$J(t) = \int_0^\infty dt\ \frac{1}{t^2}\ e^{-\left(t+\frac{z^2}{4t}\right)}$$

$$=: \int_0^\infty dt\ e^{-F(t)}$$

$$F(t) = \ln t^2 + t + \frac{z^2}{4t}$$

Find extremal value $\bar{t}$ with $F'(t') = 0$, $F''(t') > 0$ (local minimum)

$$F'(t) = \frac{2}{t} + 1 - \frac{z^2}{4}\frac{1}{t^2}$$

$$F'(\bar{t}) = 0 \quad \Rightarrow \quad \bar{t}_\pm = -1 \pm \sqrt{1+\frac{z^2}{4}}$$

For z large we find approx.: $\quad \bar{t} \approx \frac{z}{2}$

$$F''(t) = -\frac{2}{t^2} + \frac{z^2}{2}\frac{1}{t^3}$$

$$F''(\bar{t}) = -2\frac{4}{z^2} + \frac{z^2}{2}\cdot\frac{8}{z^3} = -\frac{8}{z^2} + \frac{4}{z} \approx \frac{4}{z} \qquad \text{for } z\to\infty$$

We also need

$$F(\bar{t}) = \ln\frac{z^2}{4} + \frac{z}{2} + \frac{z^2}{4}\ \frac{2}{z}$$

$$= \ln\frac{z^2}{4} + \frac{z}{2} + \frac{z}{2} = \ln\frac{z^2}{4} + z$$

Approximation :

$$J(t) = \int_0^{\infty} dt \; e^{-F(t)}$$

$$\approx \int_{-\infty}^{+\infty} dt \quad e^{-F(\bar{t}) - \frac{1}{2} F''(\bar{t}) (t - \bar{t})^2}$$

$$= e^{-F(\bar{t})} \underbrace{\int_{-\infty}^{+\infty} dt \; e^{-\frac{1}{2} F''(\bar{t}) (t - \bar{t})^2}}_{\text{Gauss integral}}$$

a) $e^{-F(\bar{t})} = e^{-\left(\ln \frac{z^2}{4} + z\right)} = \frac{4}{z^2} \cdot e^{-z}$

b) $\int_{-\infty}^{+\infty} dt \; e^{-\alpha t^2} = \sqrt{\frac{\pi}{\alpha}} = \sqrt{\frac{\pi}{2} z}$

$\alpha = \frac{2}{z}$

$\Rightarrow \quad K_1(z) \approx \frac{z}{4} \cdot \frac{4}{z^2} e^{-z} \cdot \sqrt{\frac{\pi}{2} z} = \sqrt{\frac{\pi}{2z}} e^{-z}$

O(4) breaking at $O(a^2)$: $\mathcal{L}_{break} = c\cdot a^2\, \bar{q}\, \gamma_\mu D_\mu D_\mu D_\mu\, q$

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- breaks O(4)
- preserves chiral symmetry

Map to ChPT:

Promote $c\cdot a^2$ to a tensor field $C_{\alpha\beta\gamma\delta}$ which transforms as a singlet under chiral transformations

$$C_{\alpha\beta\gamma\delta} \xrightarrow{R,L} C_{\alpha\beta\gamma\delta}$$

"Physical value": $C_{\alpha\beta\gamma\delta} = \sum_\mu \delta_{\alpha\mu}\delta_{\beta\mu}\delta_{\gamma\mu}\delta_{\delta\mu}$

↳

$$\mathcal{L}_{break} = \bar{q}\, C_{\alpha\beta\gamma\delta}\, \gamma_\alpha D_\beta D_\gamma D_\delta\, q$$

is O(4) invariant if $C_{\alpha\beta\gamma\delta}$ is an O(4) tensor, rank 4

⇒ Construct invariants with $C_{\alpha\beta\gamma\delta}$ in ChPT!

↳ We need chirally invariant terms with 4 O(4) indices!

↳

$$X_1 = \mathrm{tr}\, \partial_\alpha U\, \partial_\beta U^+\, \partial_\gamma U\, \partial_\delta U^+$$

$$X_2 = \mathrm{tr}\, \partial_\alpha U^+\, \partial_\beta U\, \partial_\gamma U^+\, \partial_\delta U$$

Note: $X_1 \xrightarrow{P} X_2$ ⇒ We need $X_1 + X_2$ to be P invariant

There are no additional independent terms:

e.g.

$$\mathrm{tr}\, \partial_\alpha U\, \partial_\beta U^+ \underbrace{U \partial_\gamma U^+}_{= -\partial_\gamma U\, U^+} U \partial_\sigma U^+ = -X_1$$

$$\mathrm{tr}\, \partial_\alpha U \underbrace{U^+ \partial_\beta U}_{-\partial_\beta U^+ U} \underbrace{\partial_\gamma U^+ U}_{-U^+\partial_\gamma U} \partial_\sigma U^+ = X_1$$

etc.

$$\Rightarrow \quad \mathcal{L}^{ChPT}_{break} = \frac{K}{2} \underset{\text{LEC}}{C_{\alpha\beta\gamma\delta}} \left[\mathrm{tr}\, \partial_\alpha U\, \partial_\beta U^+ \partial_\gamma U\, \partial_\delta U^+ + \mathrm{tr}\, \partial_\alpha U^+ \partial_\beta U\, \partial_\gamma U^+ \partial_\delta U \right]$$

Set $C_{\alpha\beta\gamma\delta}$ to physical value

$$\mathcal{L}^{ChPT}_{break} = \frac{K}{2} \cdot c\, a^2 \left(\mathrm{tr}\, \partial_\mu U\, \partial_\mu U^+ \partial_\mu U\, \partial_\mu U^+ + \mathrm{tr}\, \partial_\mu U^+ \partial_\mu U\, \partial_\mu U^+ \partial_\mu U \right)$$

same terms

$$\boxed{\mathcal{L}_{break} = K \cdot c \cdot a^2\, \mathrm{tr}\left(\partial_\mu U\, \partial_\mu U^+ \partial_\mu U\, \partial_\mu U^+ \right)} \qquad (1)$$

Expansion in pion fields: $\propto \left(\partial_\mu \pi^a \partial_\mu \pi^a\right)\left(\partial_\mu \pi^b \partial_\mu \pi^b\right)$

Power counting: If $a \sim m_q \sim p^2 \Rightarrow$ (1) is of $O(p^8)$

Very small!