Lecture 3, slide 35: Derive

$$\Delta M^2_{\pi^\pm} = \Delta M^2_{K^\mp} = \frac{2e^2C}{f^2}$$

$$\Delta M^2_{\pi^0} = \Delta M^2_{\eta} = \Delta M^2_{K^0} = 0$$

starting from

$$\mathcal{L} = e^2C \, \mathrm{tr}[Q U Q U^+] \quad , \; Q = \mathrm{diag}\left(\tfrac{2}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}\right)$$

Expand U, U^+ to quadratic order:

$$U = e^{\frac{2i}{f}\Pi} = 1 + \frac{2i}{f}\Pi - \frac{2}{f^2}\Pi^2$$

$$U^+ = e^{-\frac{2i}{f}\Pi} = 1 - \frac{2i}{f}\Pi - \frac{2}{f^2}\Pi^2$$

with U given in Lecture 2, slide 24:

$$\Pi = \frac{1}{2}\begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}$$

$$= \frac{1}{2}\left(\Pi_A + \Pi_B\right) \quad , \; \Pi_B = \mathrm{diag}\left(\pi^0 + \tfrac{1}{\sqrt{3}}\eta, -\pi^0 + \tfrac{1}{\sqrt{3}}\eta, -\tfrac{2}{\sqrt{3}}\eta\right)$$

$=$ linear comb. of T^3, T^8

$$\Pi_A = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \pi^+ & K^+ \\ \pi^- & 0 & K^0 \\ K^- & \bar{K}^0 & 0 \end{pmatrix}$$

The terms quadratic in Π in $\mathcal{L}$ are

$$\mathcal{L}_{\pi^2} = e^2C \left\{ \frac{4}{f^2} \mathrm{tr}\, Q\Pi Q\Pi - \frac{2}{f^2}\mathrm{tr}\, Q^2\Pi^2 - \frac{2}{f^2}\mathrm{tr}\, \Pi^2 Q^2 \right\}$$

$$= e^2C \frac{4}{f^2}\left\{ \mathrm{tr}\, Q\Pi Q\Pi - \mathrm{tr}\, QQ\Pi\Pi \right\}$$

Note: $Q\pi_B = \pi_B Q$ since both are elements of the Cartan sub algebra spanned by T^3, T^8

$$\Rightarrow \quad \Delta M^2_{\pi^0} = \Delta M^2_{\eta} = 0$$

For the remaining mass shifts we need

$$\operatorname{tr} Q\pi Q\pi = -\frac{2}{9}\left(\pi^+\pi^- + K^+K^-\right) + \frac{1}{9} K^0 \bar{K}^0$$

$$\operatorname{tr} Q^2\pi^2 = \frac{5}{18}\left(\pi^+\pi^- + K^+K^-\right) + \frac{1}{9} K^0 \bar{K}^0$$

$$\Rightarrow \quad \operatorname{tr} Q\pi Q\pi - \operatorname{tr} Q^2\pi^2 = -\frac{1}{2}\left(\pi^+\pi^- + K^+K^-\right)$$

$$\Rightarrow \quad \mathcal{L}_{\pi^2} = -\frac{2e^2C}{f^2}\left(\pi^+\pi^- + K^+K^-\right)$$

$$\rightsquigarrow \qquad \Delta M^2_{\pi^\pm} = \Delta M^2_{K^\pm} = \frac{2e^2C}{f^2}$$

$$\Delta M^2_{\bar{K}^0} = \Delta M^2_{K^0} = 0$$

see Lect 2, slide 22:
Convince yourself that the chiral condensate in massless QCD (ChPT) does not receive pion loop corrections

Slide 22: $$-\langle 0|\bar{q}q|0\rangle = \frac{1}{i}\frac{\partial}{\partial m_u}\langle 0|e^{i\int d^4x\,\mathcal{L}_{ChPT}}|0\rangle\Big|_{m_q=0}$$

$$\frac{\partial}{\partial m_u}\mathcal{L}_2^m = \frac{f^2B}{2}\frac{\partial}{\partial m_u}\operatorname{tr}\mathcal{M}(U+U^+)$$

$$U+U^+ = 1+\frac{2i}{f}\pi - \frac{2}{f^2}\pi^2 + \dots$$
$$+1-\frac{2i}{f}\pi - \frac{2}{f^2}\pi^2 + \dots$$
$$= 2-\frac{4}{f^2}\pi^2$$

Consider SU(2) ChPT with $m_u = m_d = m$ for simplicity

$$\rightsquigarrow \frac{\partial}{\partial m_u}\operatorname{tr}\mathcal{M}(U+U^+) = \frac{\partial}{\partial m}\, m\operatorname{tr}\left(2-\frac{4}{f^2}\pi^2\right)$$
$$= 4-\frac{4}{f^2}\operatorname{tr}\pi^2$$
$$= 4-\frac{2}{f^2}\pi^a\pi^a$$

$$\Rightarrow \langle 0|4-\frac{2}{f^2}\pi^a(x)\pi^a(x)|0\rangle = 4-\frac{2}{f^2}\underbrace{\langle 0|\pi^a(x)\pi^a(x)|0\rangle}$$

π-prop. at zero distance

$$G(0) = \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-i\varepsilon}$$

massless QCD!

In Dim Reg. this leads to

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - i\varepsilon} \xrightarrow[\text{Wick}]{\text{D dim}} A_0(0) = 0$$

No 1-loop correction to the chiral condensate!

Expanding U, U^+ to more π fields leads to products of π-propagators, thus these corrections also vanish!