QCD in finite volume

Sasa Prelovsek

University of Ljubljana

Jozef Stefan Institute, Ljubljana, Slovenia

University of Regensburg

EuroPLEx School, fall 2020

Lecture 3

Outline, lecture 3

- interpolators
- Wick contractions
- all-to-all propagators: distillation method
- Applications:

- bound states
- virtual bound states
- resonances (covered in lecture 4)

In all cases : energy regions where a single channel H1 H2 is present

Interpolators

they create/annihilate states of the system with given quantum number (irrep / J^{PC})

meson example (of course quarks q can have different flavors)

$$\bar{q}(0)\ \Gamma D_i D_j\ q(0)$$

$$\sum_{\vec{x}} \bar{q}(\vec{x})\ \Gamma D_i D_j\ q(\vec{x})\ e^{i\vec{P}\vec{x}}$$

$$\sum_j \qquad M_1(\vec{p}_{1j}) \qquad\qquad M_2(\vec{p}_{2j})$$

$$\sum_j \sum_{\vec{x}} \bar{q}(\vec{x})\Gamma_{1j} q(\vec{x}) e^{i\vec{p}_{1j}\vec{x}} \sum_{\vec{y}} \bar{q}(\vec{y})\Gamma_{2j} q(\vec{y}) e^{i\vec{p}_{2j}\vec{y}}$$

$$\sum_{\vec{x}} [\bar{q}(\vec{x})\Gamma_1 \bar{q}(\vec{x})]_{3_c} [q(\vec{x})\Gamma_2 q(\vec{x})]_{\bar{3}_c}\ e^{i\vec{P}\vec{x}}$$

Wick contractions: all-to-all propagators needed

example of M1 M2 <-> M1 M2 contraction

$O(t_f)$ $O(t_i)$

$$\sum_j \underbrace{\sum_{\vec{x}_f} \bar{q}(\vec{x}_f,t_f)\Gamma_{1j}q(\vec{x}_f,t_f)e^{i\vec{p}_{1j}\vec{x}_f}}_{M_1(\vec{p}_{1j},t_f)} \underbrace{\sum_{\vec{y}_f} \bar{q}(\vec{y}_f,t_f)\Gamma_{2j}q(\vec{y}_f,t_f)e^{i\vec{p}_{2j}\vec{y}_f}}_{M_2(\vec{p}_{2j},t_f)} \qquad \sum_j \underbrace{\sum_{\vec{x}_i} \bar{q}(\vec{x}_i,t_i)\Gamma_{1j}q(\vec{x}_i,t_i)e^{i\vec{p}_{1j}\vec{x}_i}}_{M_1(\vec{p}_{1j},t_i)} \underbrace{\sum_{\vec{y}_i} \bar{q}(\vec{y}_i,t_i)\Gamma_{2j}q(\vec{y}_i,t_i)e^{i\vec{p}_{2j}\vec{y}_i}}_{M_2(\vec{p}_{2j},t_i)}$$

$$q(\vec{x}_f,t_f)\bar{q}(\vec{x}_i,t_i)$$

Propagators from all points x_i, t_i to all points x_f,t_f needed

$$\sum_j \underbrace{\sum_{\vec{x}_f} \bar{q}(\vec{x}_f,t_f)\Gamma_{1j}q(\vec{x}_f,t_f)e^{i\vec{p}_{1j}\vec{x}_f}}_{M_1(\vec{p}_{1j},t_f)} \underbrace{\sum_{\vec{y}_f} \bar{q}(\vec{y}_f,t_f)\Gamma_{2j}q(\vec{y}_f,t_f)e^{i\vec{p}_{2j}\vec{y}_f}}_{M_2(\vec{p}_{2j},t_f)} \qquad \sum_j \underbrace{\sum_{\vec{x}_i} \bar{q}(\vec{x}_i,t_i)\Gamma_{1j}q(\vec{x}_i,t_i)e^{i\vec{p}_{1j}\vec{x}_i}}_{M_1(\vec{p}_{1j},t_i)} \underbrace{\sum_{\vec{y}_i} \bar{q}(\vec{y}_i,t_i)\Gamma_{2j}q(\vec{y}_i,t_i)e^{i\vec{p}_{2j}\vec{y}_i}}_{M_2(\vec{p}_{2j},t_i)}$$

if those two are of same flavor

$$q(\vec{x}_f,t_f)\bar{q}(\vec{y}_f,t_f)$$

Propagators from all points y_f, t_f to all points x_f,t_f needed

Brute force: this would require calculation propagators from all (x,t) space time points: N_L^3 x N_T x 3 inversions.
(for each Dirac index)

In practice: this is not feasible

Distillation method for all-to-all propagators

most-weidely employed method to get all-to-all propagators:

distillation method: Peardon et al 2009 *Phys.Rev.D* 80 (2009) 054506

I suggest to read the original reference to learn the method

$$q(\vec{x},t) \;\rightarrow\; q_s(\vec{x},t)$$

spatially-smeared quark around point x

$$q_s^{\alpha c}(\vec{x},t) = \sum_{k=1}^{k=N_v} \underbrace{v_{\vec{x}c}^{(k)}(t) v_{\vec{x}'c'}^{(k)\dagger}(t)\; q^{\alpha c'}(\vec{x}',t)}_{\text{Laplace Heaviside smearing (cuts-out higher frequency modes)}}$$

spectral decomposition

$v^{(k)}(t)$ k=1,...,N are eigenvectors of lattice Laplace-operator(t) which is NxN matrix, $N=3\,N_L^3$

if Nv=N : $q_s=q$, Nv=O(100) << $3\,N_L^3$ is taken

Distillation method for all-to-all propagators

number of inversions per Dirac index

brute force: propagators from all x_i, t_i to all x_f, t_f

$3 \times N_L^3 \times N_T$

(not feasible)

$$q(\vec{x}_f, t_f)\bar{q}(\vec{x}_i, t_i)$$

distillation : propagators from $v^{(ki)}(t_i)$ (ki=1,..,Nv) to $v^{(kf)}(t_f)$ (kf=1,..,Nv)

$3 \times N_v$, $N_v \sim O(100)$

$$\tau^{k_f k_i}(t_f, t_i) \equiv v^{(k_f)}(t_f) v^{(k_i)\dagger}(t_i)$$

these propagators are calculated and stored

they are called peramulators

all Wick contractions can be expressed in terms of them

(as long as one is happy with using smeared quarks)

for matrix elements of local currents one needs point-quarks $\langle H_f | \bar{q}(x) \Gamma \bar{q}(x) | H_i \rangle$

Reminder how to identify on the lattice bound state, virtual bound state, resonances

$$T = \frac{1}{p \cot \delta_l - ip}$$

Scattering of two hadrons H1 H2

- assuming that only one channel is present in the given energy region (elastic scattering)
- H1 H2 carry no spin

- Determine eigen-energies E , $E=E_{cm}$
- Determine $T_l(E)$ for these E via Luscher's equation

Riemann sheet I if Im(p)>0 , Riemann sheet II if Im(p)<0 $E=\sqrt{m_1^2+p^2}+\sqrt{m_2^2+p^2}$

- location of pole in the scattering amplitude T(E) help to identify

bound state: pole for real E below threshold Riemann sheet I

virtual bound state: pole for real E below threshold Riemann sheet II

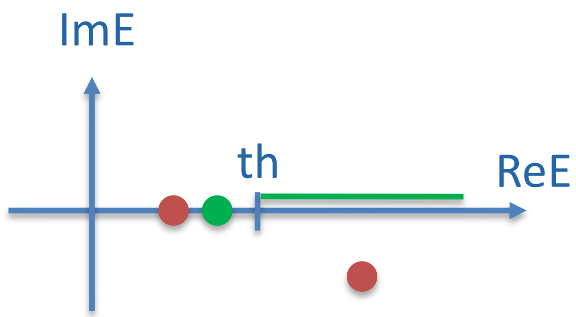
no need to cosider complex E for (virtual) bound states

- for resonances : T (real E) -> T(complex E)

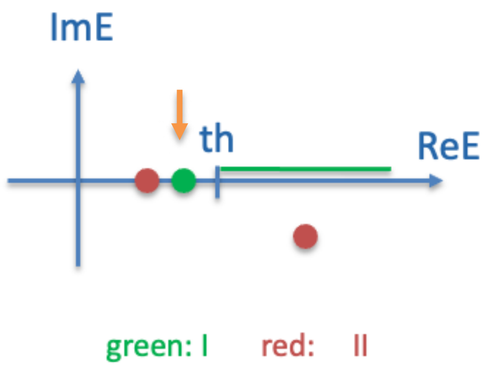
resonance: pole away from real axes E=m-1/2 i Γ Riemann sheet II

these poles affect physical scattering ("peaks in σ")

if they are close to physical axes (green line) Riemann sheet I



green: I red: II



Shallow bound states

pole of T(E) for real E below threshold

p=i|p| : Riemann sheet I

D_{s0}^* shallow bound state in DK scattering: I=0, $J^P=0^+$

D_{s0}^* lies 45 MeV below DK strong decay threshol in experiment

Mode		Fraction (Γ_i /Γ)
Γ_1	$D_s^+\gamma$	$(93.5 \pm 0.7)\%$
Γ_2	$D_s^+\pi^0$	$(5.8 \pm 0.7)\%$
Γ_3	$D_s^+ e^+ e^-$	$(6.7 \pm 1.6) \times 10^{-3}$

D. Mohler, C. Lang, L. Leskovec, S.P. , R. Woloshyn: Phys. Rev. Lett. 2013, PRD 2014

m_π =156 MeV, Nf=2+1, V=32^3x64, L=2.9 fm [PACS-CS] PACSCS

P_{tot} =0 : interpolators in irrep A_1^+ of Oh contain J=0 states

$O^{qq} = \bar{s}c$ (p=0)

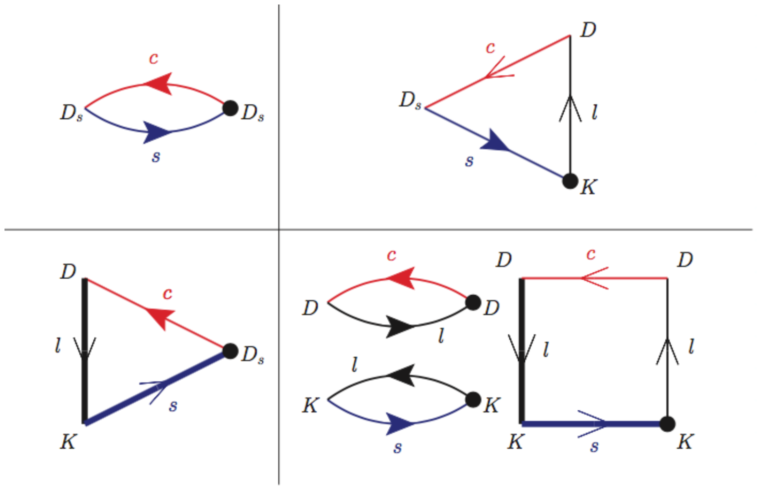
$\bar{s}\gamma_i\nabla_i c$

$\bar{s}\gamma_t\gamma_i\nabla_i c$

$\bar{s}\nabla_i\nabla_i c$

$$O_1^{DK} = [\bar{s}\gamma_5 u]\,(\vec{p}=0)\,[\bar{u}\gamma_5 c]\,(\vec{p}=0) + \{u \to d\}\ ,$$
$$O_2^{DK} = [\bar{s}\gamma_t\gamma_5 u]\,(\vec{p}=0)\,[\bar{u}\gamma_t\gamma_5 c]\,(\vec{p}=0) + \{u \to d\}$$
$$O_3^{DK} = \sum_{\vec{p}=\pm e_{x,y,z}\ 2\pi/L} [\bar{s}\gamma_5 u]\,(\vec{p})\,[\bar{u}\gamma_5 c]\,(-\vec{p}) + \{u \to d\}\ .$$

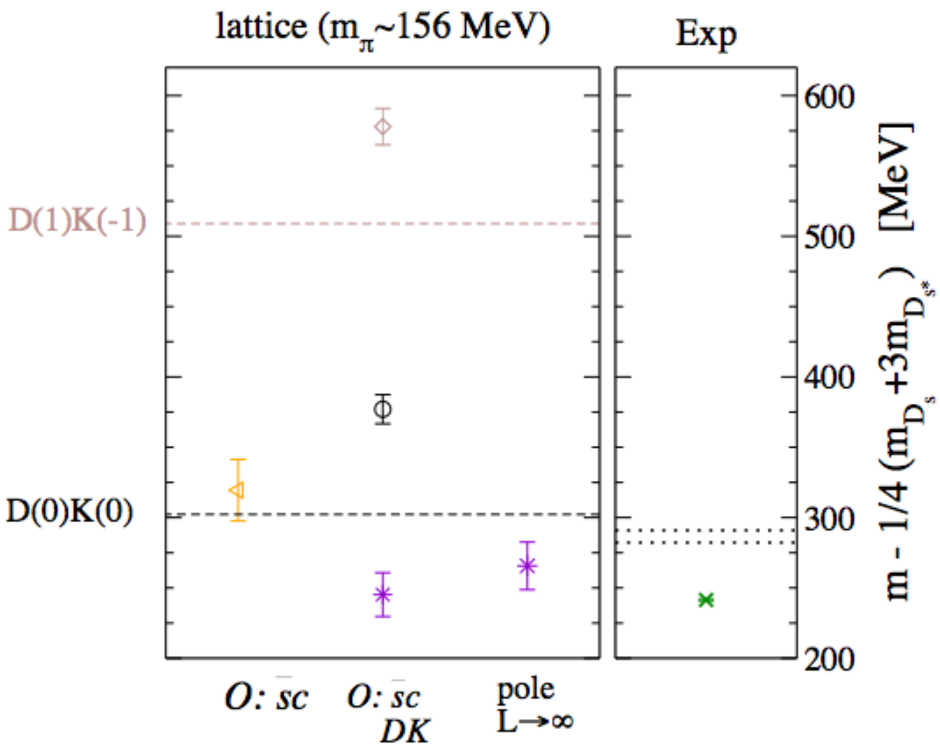
$$C_{ij}(t) = \langle 0|O_i(t)\,O_j^\dagger|0\rangle = \sum_k Z_{ik}^\dagger Z_{kj} e^{-E_k t}$$



C evaluated using distillation method [Peardon et al.]

Figure by RQCD

$D_{s0}^*(2317)$



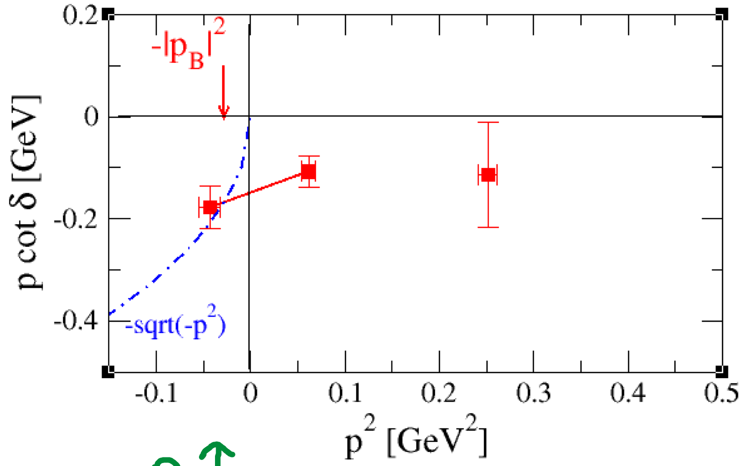
D. Mohler, C. Lang, L. Leskovec, S.P. , R. Woloshyn:

Phys. Rev. Lett. 2013: $m_\pi \approx 156$ MeV, L≈2.9 fm, Nf=2+1, PACSCS

mesonic bound st. established on lattice for the first time

$D_{s0}^*(2317)$	m - ¼ (m_{Ds}+3m_{Ds^*})	mD+mK-m
lat	266 ± 16±4 MeV	36 ± 17 MeV
exp	241.45 ± 0.6 MeV	45 MeV

$$E = \sqrt{m_D^2 + p^2} + \sqrt{m_K^2 + p^2} \qquad p\cot\delta(p) = \frac{2}{\sqrt{\pi}L} Z_{00}(1, (\tfrac{pL}{2\pi})^2)$$



Luscher 1991

$p\cot\delta$

$i\,p = i\,i\,|p| =$

$-\sqrt{-p^2}$

B ↑

$$p\cot\delta(p) = \frac{1}{a_0} + \frac{1}{2} r_0 p^2$$

$$a_0 = -1.33 \pm 0.20 \text{ fm} \quad r_0 = 0.27 \pm 0.17 \ fm$$

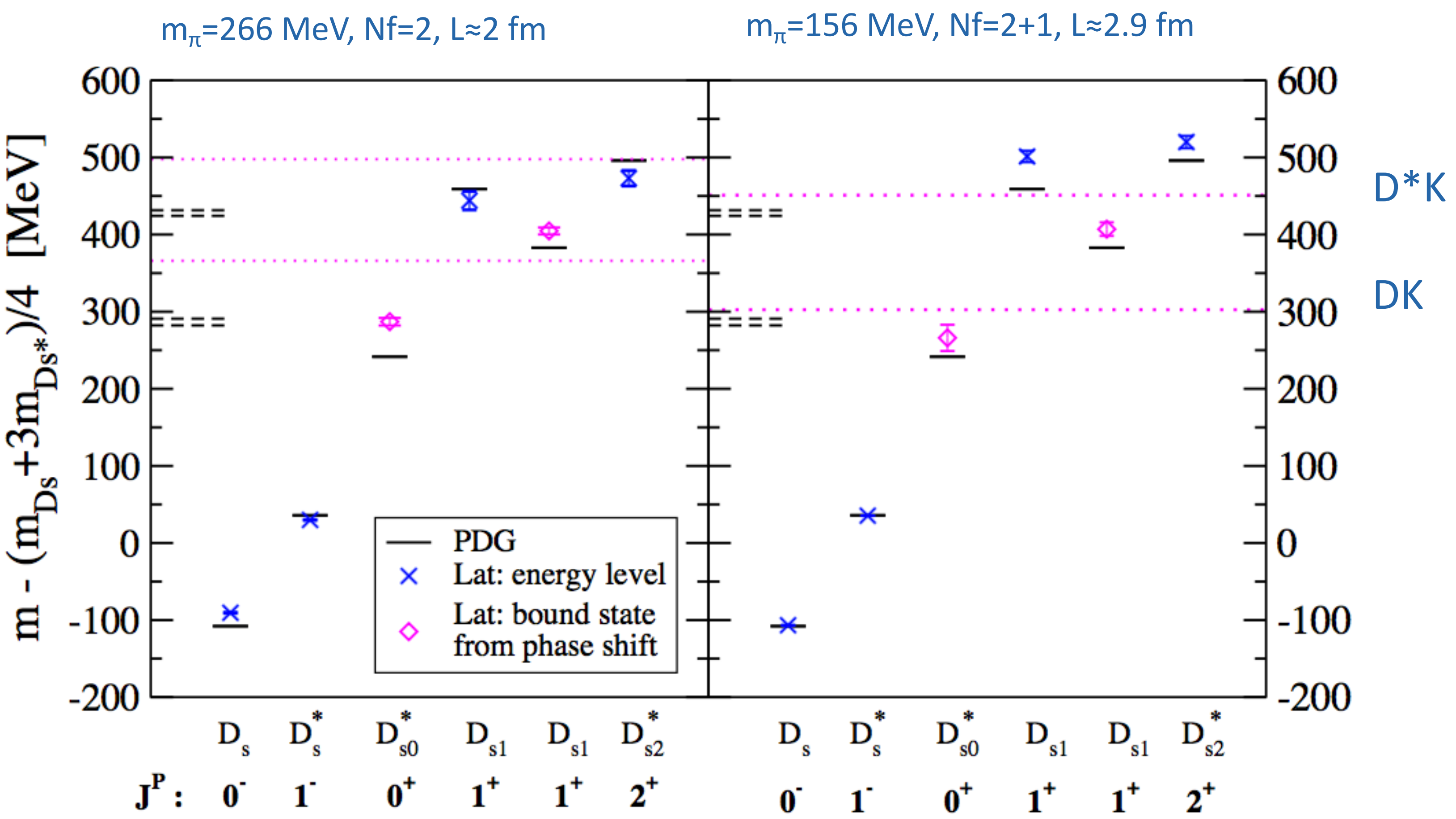
$$\boxed{T = \frac{1}{p\cot\delta_0 - ip}}$$

$$ip_B = p_B \cot\delta(p_B)\ , \quad p_B = i|p_B|$$

$$i\,|p_B|*i = \frac{1}{a_0} - \frac{1}{2} r_0 |p_B|^2 \quad \rightarrow \quad |p_B|^2 = 0.028 \pm 0.012\ GeV^2$$

$$m_{D_{s0}}^{lat, L\to\infty} = \sqrt{m_D^2 - |p_B|^2} + \sqrt{m_K^2 - |p_B|^2}$$

Spectrum of Ds mesons



Lang, Leskovec, Mohler, S.P. , Woloshyn: PRD 2014, Phys. Rev. Lett. 2013

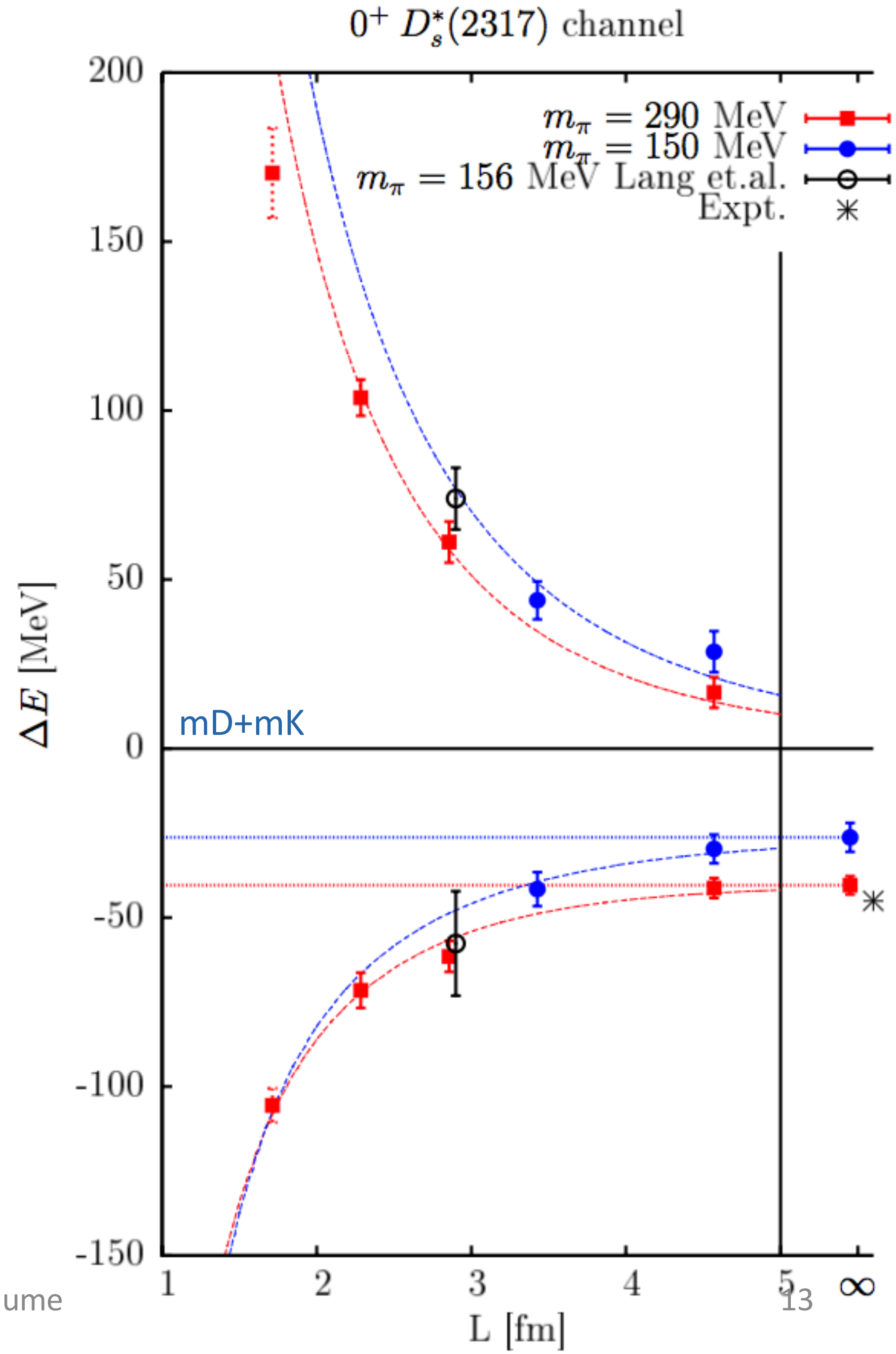
$D_{s0}^*(2317)$

Bali, Collins, Cox, Schafer (RQCD):

PRD (2017) 074501

κ_l	a [fm]	V	am_π	m_π [MeV]
0.13632	0.071	$24^3 \times 48$	0.1112(9)	306.9(2.5)
	0.071	$32^3 \times 64$	0.10675(52)	294.6(1.4)
	0.071	$40^3 \times 64$	0.10465(38)	288.8(1.1)
	0.071	$64^3 \times 64$	0.10487(24)	289.5(0.7)
0.13640	0.071	$48^3 \times 64$	0.05786(55)	159.7(1.5)
	0.071	$64^3 \times 64$	0.05425(49)	149.7(1.4)

$$O: \quad \bar{s}c,\ D(0)K(0),\ \cancel{D(1)K(-1)}$$

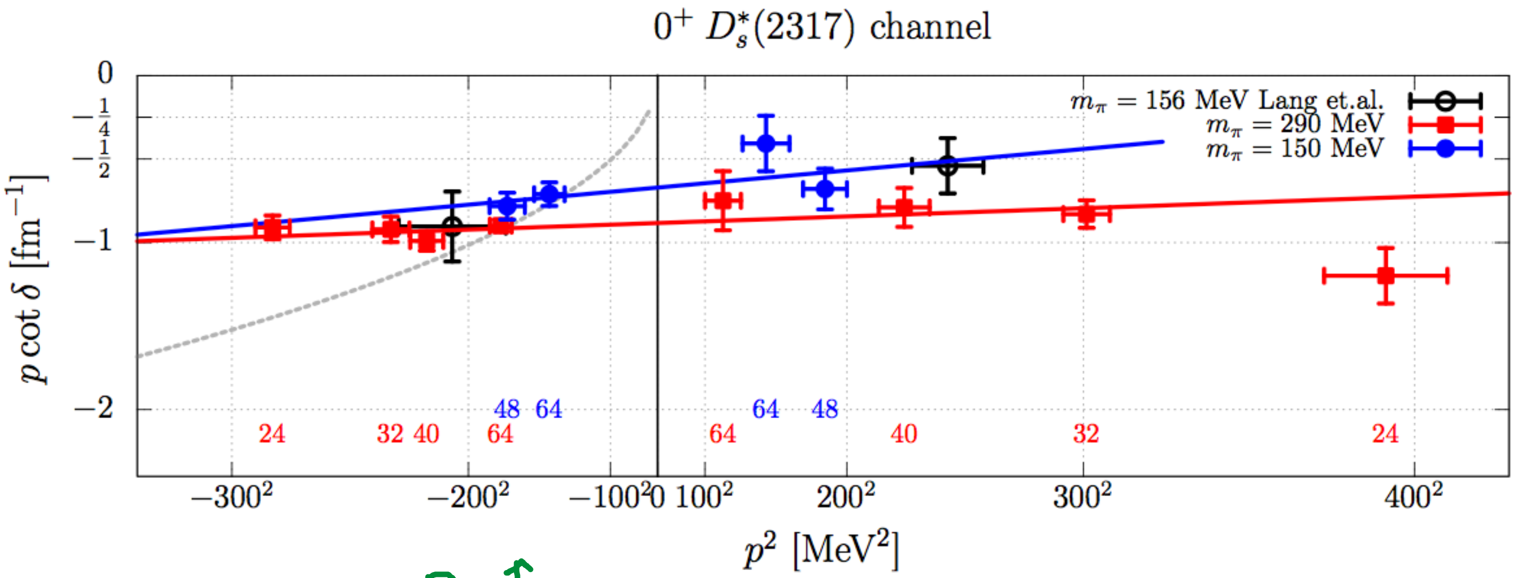


$D_{s0}^*(2317)$

Bali, Collins, Cox, Schafer (RQCD):

PRD (2017) 074501

$$O: \quad \bar{s}c,\ D(0)K(0),\ \cancel{D(1)K(-1)}$$



$$p\cot\delta(p) = \frac{1}{a_0} + \frac{1}{2} r_0 p^2$$

B ↑

0^+ channel

	$m_\pi = 290$ MeV	$m_\pi = 150$ MeV	Expt.	m_π=156 MeV
a_0 [fm]	$-1.13(0.04)(+0.05)$	$-1.49(0.13)(-0.30)$		-1.33(20)
r_0 [fm]	$0.08(0.03)(+0.08)$	$0.20(0.09)(+0.31)$		0.27(17)
$\lvert p_B \rvert$ [MeV]	$180(6)(0)$	$142(11)(-9)$		
Δm [MeV]	$40(3)(0)$	$26(4)(-3)$	$42.6(0.7)(2.0)$	36 (17)
m_{D_s} [MeV]	$2384(2)(-1)$	$2348(4)(+6)$	$2317.7(0.6)(2.0)$	

RQCD

Lang et al.

Intermezzo

The need for systems with nonzero total momentum P

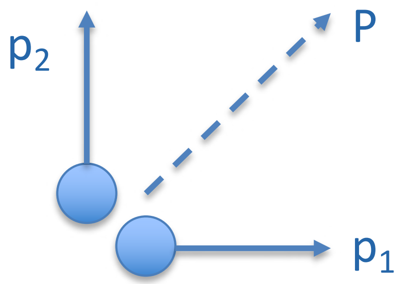
So far we considered only P=0: this renders only $T(E_{cm})$ at few values of E_{cm}

in the non-interacting limit one can reach the following E_{cm} of H_1H_2

P=0 $$E_{cm}=\sqrt{p^2+m_1^2}+\sqrt{(-p)^2+m_2^2},\quad \vec{p}=\vec{n}\frac{2\pi}{L}$$

general P $$E=\sqrt{\vec{p}_1^2+m_1^2}+\sqrt{(\vec{P}-\vec{p}_1)^2+m_2^2},\quad \vec{p}_1=\vec{n}\frac{2\pi}{L},\ \vec{P}=\vec{d}\frac{2\pi}{L}$$

$$E_{cm}=\sqrt{E^2-\vec{P}^2}$$



typical P considered:

P=(0,0,1), (1,1,0), (1,1,1) ...

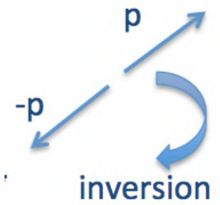
these give access to

additional E_{cm} and additional $T(E_{cm})$

Symmetries are significantly reduced for P$\neq$ 0

continuum

Parity: NOT good



Rotations/reflections:

transformations that leave P invariant

rotations around P; little group U(1)

single-particle with momentum P and spin S:

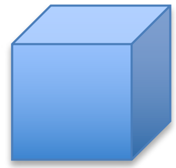
spin S of a particle: not good

only J=S+L : good

helicity : good $\qquad \lambda = \dfrac{\vec{S}\cdot\vec{P}}{|\vec{P}|}$

cubic lattice

Parity: NOT good



Rotations/reflections:

transformations that leave box and P invariant: Not much symmetry left !!

(or symmetries of box seen in moving frame)

P=(0,1,0): 8 elements

Irrep	I	$R(\pi)$	$R(3\pi/2)$	$R(\pi/2)$	Π	$R(\pi)\Pi$	$R(\pi/2)\Pi$	$R(3\pi/2)\Pi$

P=(1,1,0); 4 elements

Irrep	I	$R(\pi)$	Π	$R(\pi)\Pi$

irreps: good quantum numbers

helicity: not good

challenge: certain irrep gets contribution from both parities and several partial waves

Π is reflection in a plane that contains p; it preserves p

Luscher's relation for P= 0

$$\det\left[1 + i\mathcal{M}(E)\mathcal{G}(E)\right] = 0,$$

E=E_{cm}

$$\mathcal{G}_{l_1,m_1;l_2,m_2}(E) = \frac{p}{8\pi E}\left[\delta_{l_1,l_2}\delta_{m_1,m_2} + i\sum_{l,m}\frac{(4\pi)^2}{p^{l+1}L^3}\left(\frac{2\pi}{L}\right)^{l-2} Z_{lm}\left(1;\left(\frac{pL}{2\pi}\right)^2\right)\int d\Omega\, Y^*_{l_1,m_1}Y^*_{l,m}Y_{l_2,m_2}\right]$$

Generalization of Luscher's relation for P$\neq$ 0

m1=m2: Rummikainen, Gottlieb 1995 [hep-lat/9503028], Kim, Sachrajda, Sharpe 2005 [hep-lat/0507006]

m1$\neq m2$: Leskovec & S.P. [1202.2145], Briceno 1401.3312

$$\det\left[1 + i\mathcal{M}(E_{cm})\mathcal{G}(E_{cm})\right] = 0$$

p=momentum in cm frame

$$\mathcal{G}_{l_1,m_1;l_2,m_2}(E_{cm}) = \frac{p}{8\pi E_{cm}}\left[\delta_{l_1,l_2}\delta_{m_1,m_2} + i\sum_{l,m}\frac{(4\pi)^2}{\gamma\, p^{l+1}L^3}\left(\frac{2\pi}{L}\right)^{l-2} Z^{\vec{d}}_{lm}\left(1;\left(\frac{pL}{2\pi}\right)^2\right)\int d\Omega\, Y^*_{l_1,m_1}Y^*_{l,m}Y_{l_2,m_2}\right]$$

$$M_l(E) \equiv 8\pi E\; T_l(E) \qquad T_l = \frac{1}{p\cot\delta_l - ip} \qquad Z^{\vec{d}}_{lm}(1,q^2) \equiv \sum_{\vec{r}\in P_{\vec{d}}}\frac{r^l\, Y_{lm}(\vec{r})}{(\vec{r}^2 - q^2)}$$

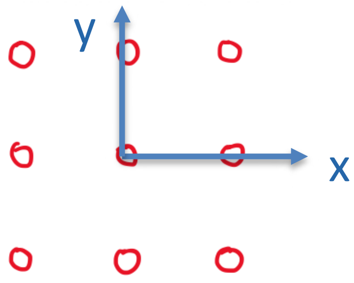
P_d: mesh of lattice points seen in cmf:

r are not integers

$$Z_{lm}^{\vec{d}}(1,q^2) \equiv \sum_{\vec{r}\in P_{\vec{d}}} \frac{r^l\ Y_{lm}(\vec{r})}{(\vec{r}^2 - q^2)}$$

$$\vec{r} = \vec{n} \in N^3$$

Symmetries for P=0: O_h

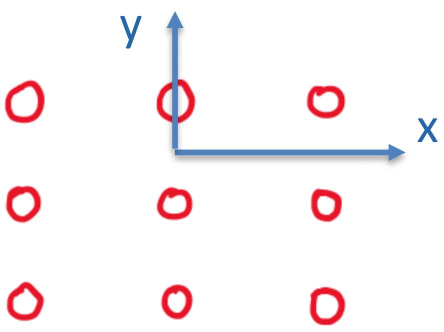


$$\vec{r}\ \frac{2\pi}{L} = \vec{p} = \hat{\gamma}^{-1}(\vec{p}_1 - \tfrac{1}{2}A\vec{P})$$

$$\vec{p}_1 = \vec{n}\tfrac{2\pi}{L}\ , \quad A = 1 + \tfrac{m_1^2 - m_2^2}{E_{cm}^2}$$

Symmetries for P ≠0

P=(0,1,0) m1=m2



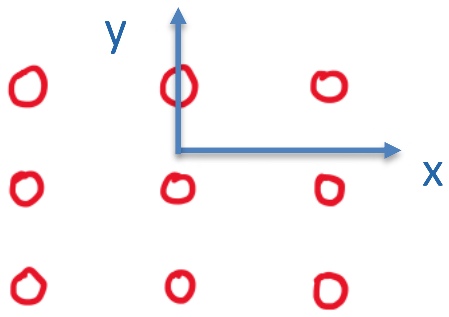
mesh Pd invariant under inversion

irrep gets contribution of a certain parity

irrep gits contribution of only even l

or only odd l

P=(0,1,0) m1 ≠ m2



both mashes are Lorentz

contracted in y-direction

mesh Pd invariant NOT under inversion

irrep gets contribution of both parities (in general, there are fortunately exceptions)

irrep gets contribution of even l AND odd l (in general, there are fortunately exceptions)

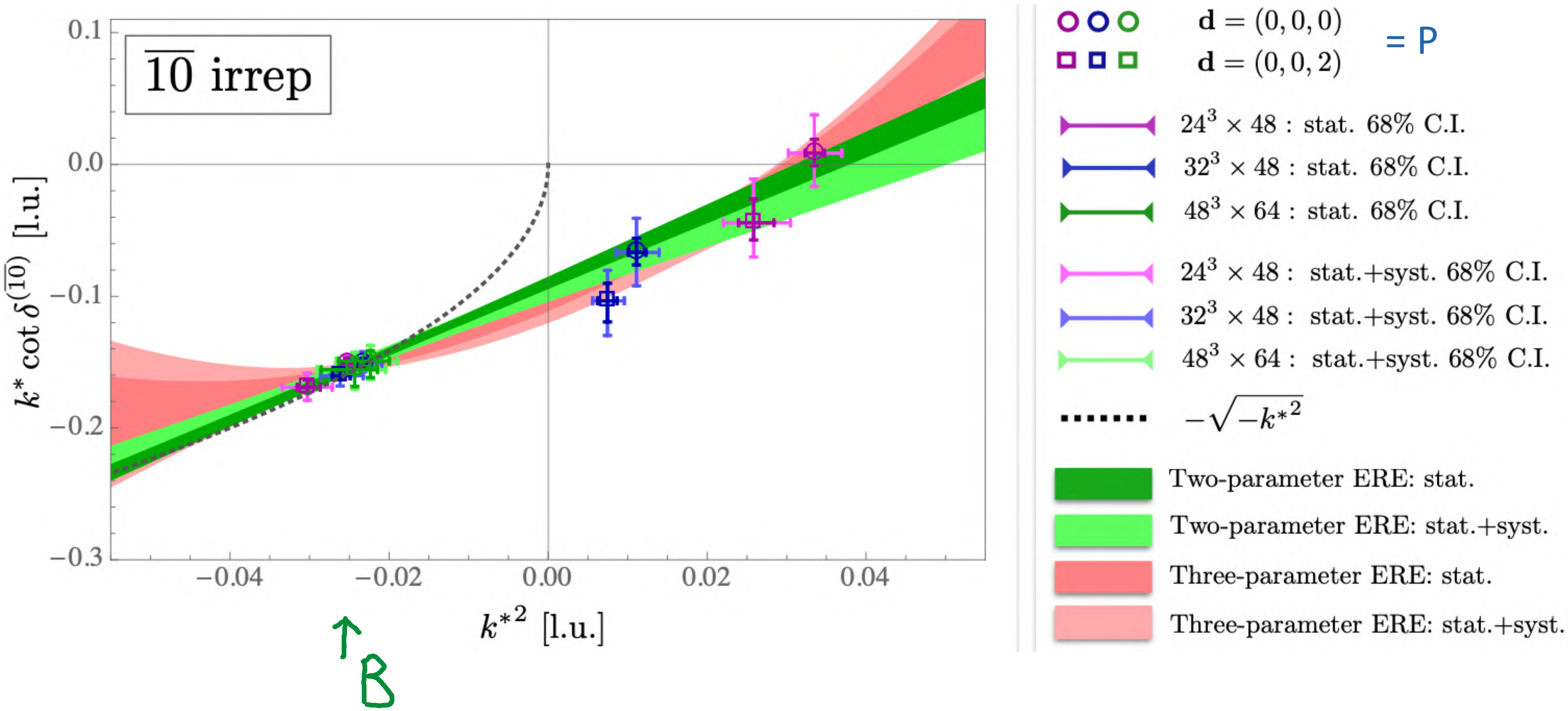
back to shallow bound states

now with simulations with P=0 and P $\neq$ 0

Deuteron (pn) channel at $m_u=m_d=m_s$

p,n carry spin: more on scattering in such case in lecture 4

NPLQCD, 1706.0655: $m_\pi \approx 800\mathrm{MeV} : \mathrm{m-mp-mn} = -28(4)\mathrm{MeV}$

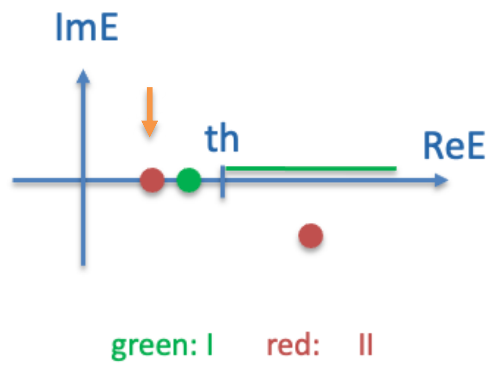


See also results by other groups: HALQCD, Mainz, Callat,Hortz et al,...

lattice groups find inconsistent results (even for SU(3) symmetric case):

some find bound state and some not

exp: $\mathrm{m-mp-mn} = -2\ \mathrm{MeV}$



Virtual bound states

pole of T(E) for real E below threshold

p= - i|p| : Riemann sheet II

These are not proper normalizable states

They are features of interaction

Dineutron (nn) channel at $m_u=m_d=m_s$

n carries spin: more on scattering in such case in lecture 4

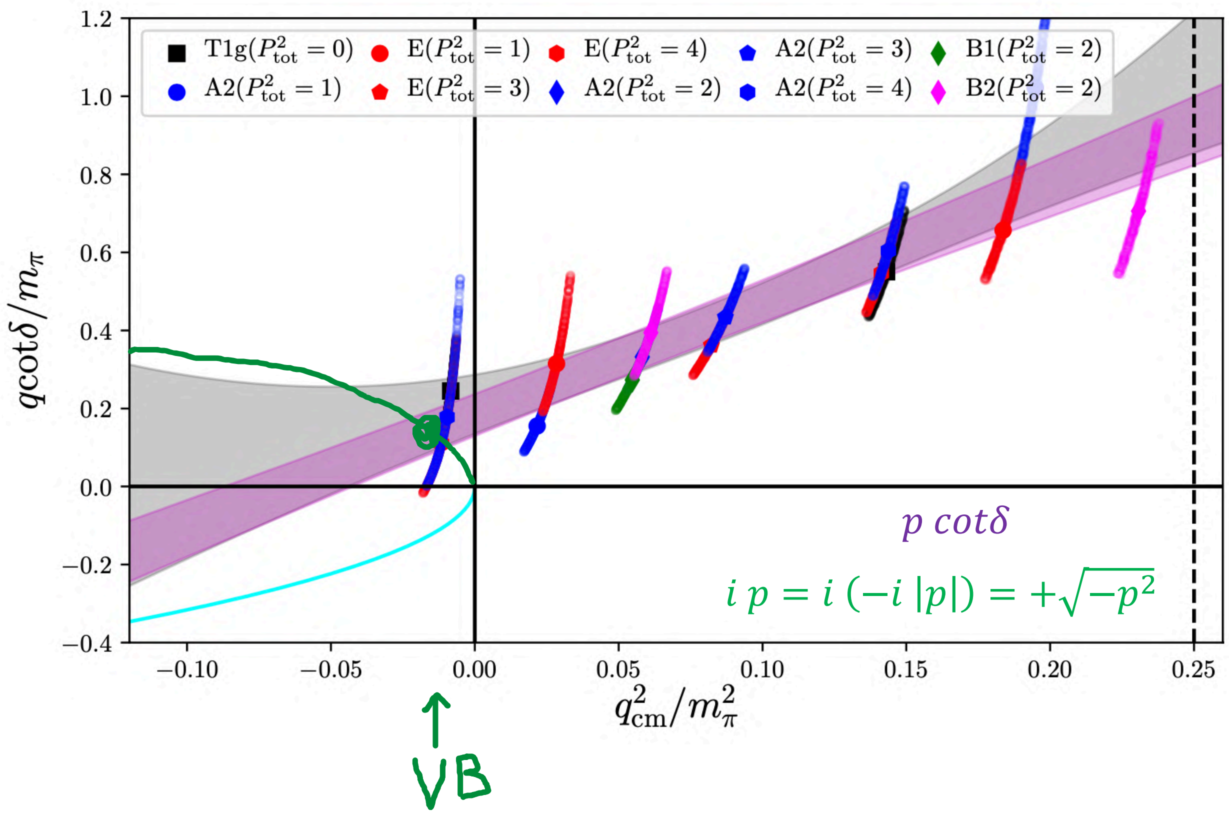
Hortz et al, 2009:11825 $m_\pi \approx 714$ MeV

$$T = \frac{1}{p\cot\delta_0 - ip}$$

$$ip = p\cot\delta_0 \;, p = -i|p|$$

$$i(-i|p|) = \frac{1}{a_0} - \frac{1}{2}r_0|p|^2$$

$$m_B = 2\sqrt{m_n^2 - |p|^2} = ?$$



exp: m − 2mn = −60 keV

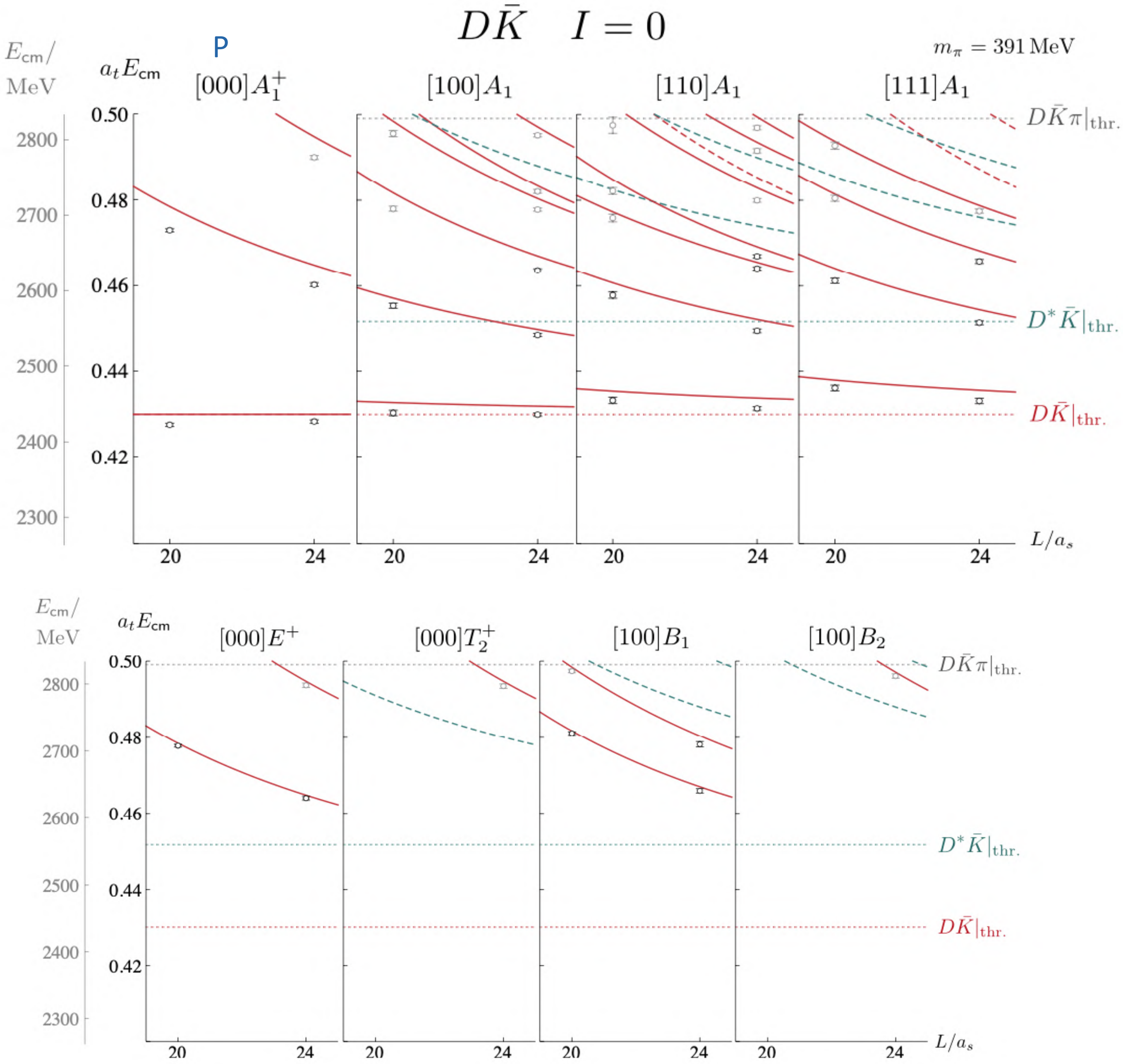
$$p\cot\delta(p) = \frac{1}{a_0} + \frac{1}{2}r_0 p^2$$

See also results by other groups: HALQCD, Mainz, Callat,Hortz et al,...

lattice groups find inconsistent results (even for SU(3) symmetric case): some find virtual bound state and some find bound state

D⁻ K⁺ $\bar{c}d\ \bar{s}u$ explicitly exotic flavor

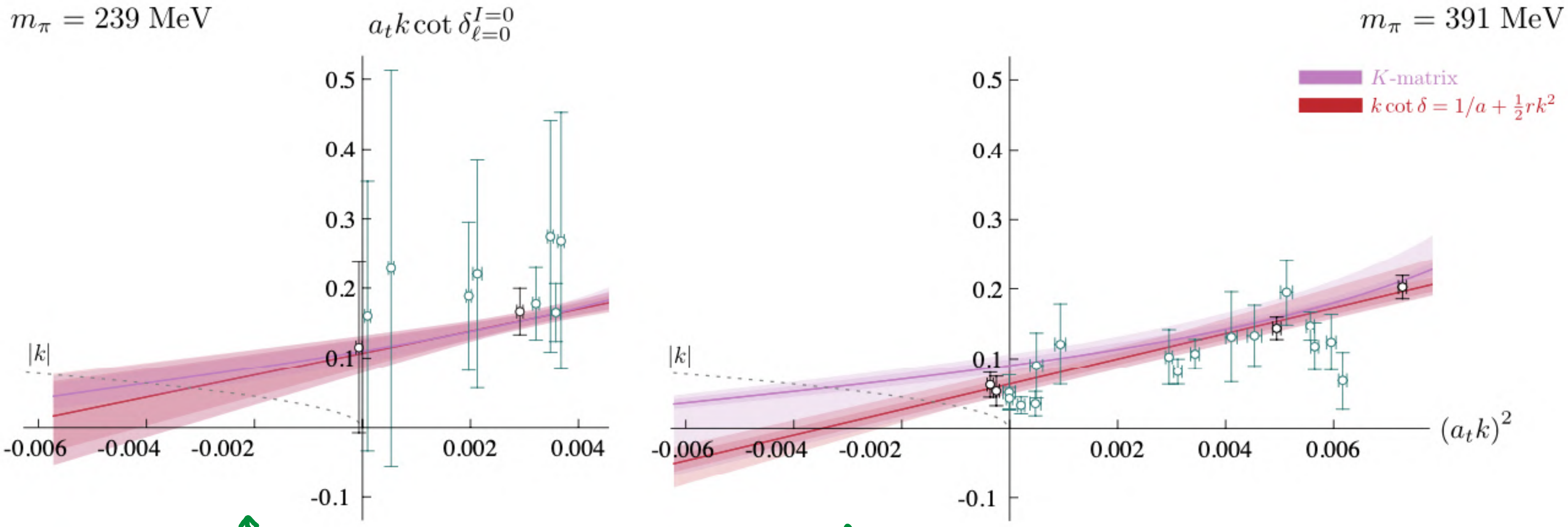
HadSpec. Coll, 2008.06432



certain irrep gets contribution fom multiple partial waves (odd and even) for P ≠ 0 since mD ≠mK

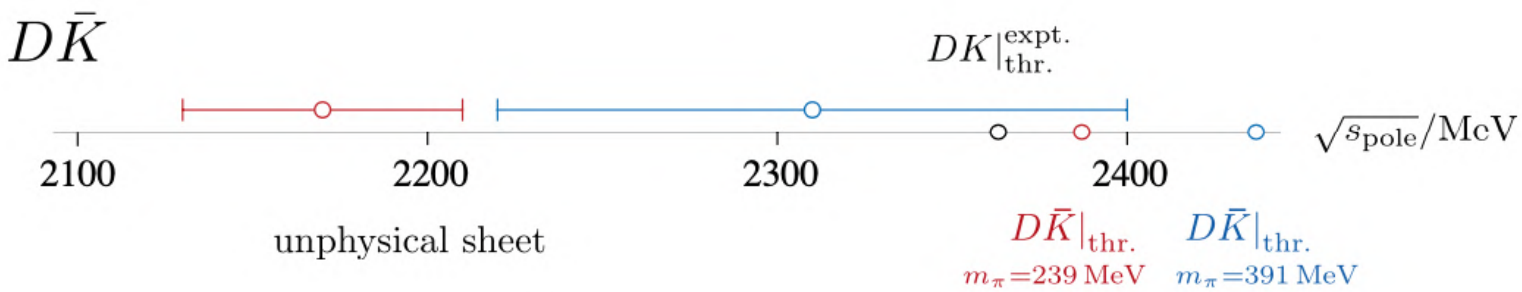
$D^- K^+$, $J^P=0^+$, I=0

HadSpec. Coll, 2008.06432



VB

VB

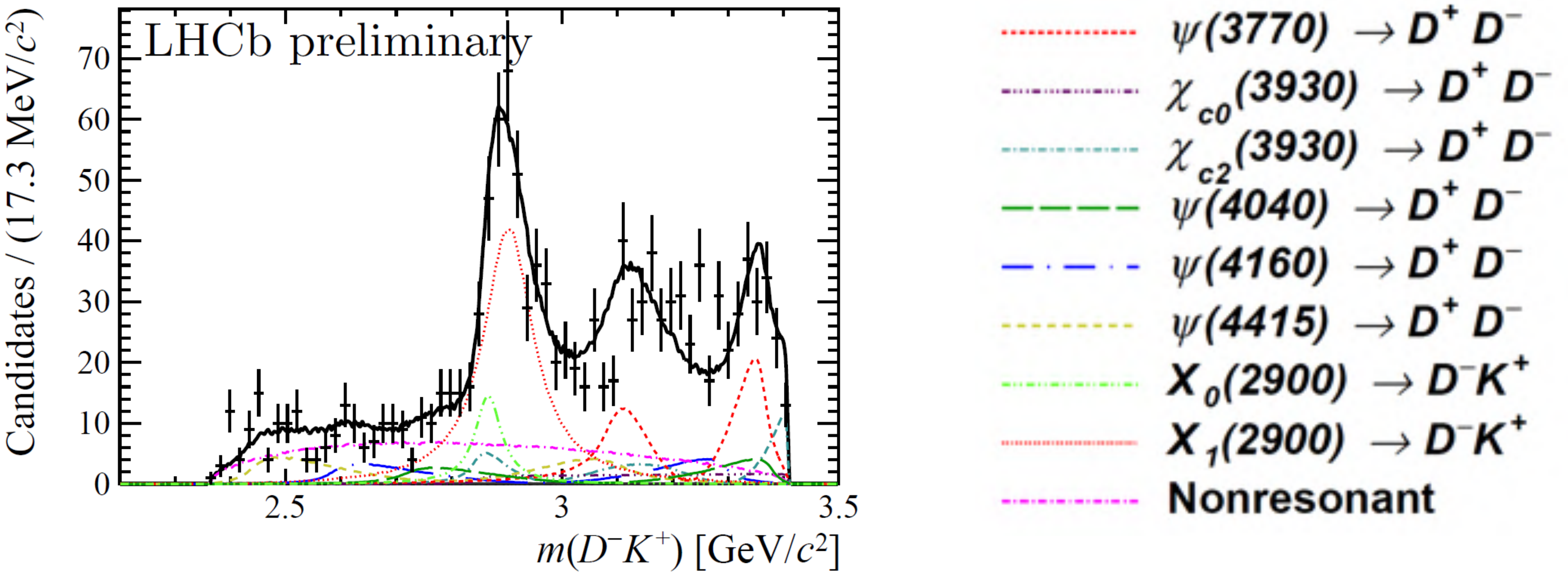


$D^- K^+$ exp evidence for two exotic resonances

LHCb
2009.00025, 2009.00026

$\bar{c}d\ \bar{s}u$

explicitly exotic flavor, isospin from exp: 0 or 1 ?



Resonance	Mass (GeV/c^2)	Width (MeV)
$X_0(2900)$	$2.866 \pm 0.007 \pm 0.002$	$57 \pm 12 \pm 4$
$X_1(2900)$	$2.904 \pm 0.005 \pm 0.001$	$110 \pm 11 \pm 4$

near D*K* thr.

energy E_{cm} =2.9 GeV

not reached by HSC lattice sim.