Introduction to Chiral Perturbation Theory

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Summary of last week

- ChPT is an expansion in powers of external pion momenta and quark masses

  dimless expansion parameters: $\frac{p^2}{(4\pi f)^2} \quad \frac{M_\pi^2}{(4\pi f)^2} \quad \frac{M_K^2}{(4\pi f)^2}$

- At a given *chiral dimension* $D = 2N+2$ only a finite number of
  - loops ($L \leq N$)
  - and vertices of $\mathcal{L}_d$ with $d \leq 2N+2$

  contribute.

	tree	1-loop	2-loop
LO	$\mathcal{L}_2$		
NLO	$\mathcal{L}_4$	$\mathcal{L}_2$	
NNLO	$\mathcal{L}_6$	$\mathcal{L}_4$	$\mathcal{L}_2$

- At a given *chiral dimension* $D = 2N+2$ only a finite number of LECs contribute
  - These are sufficient to renormalise the theory and render the theory finite
    The chiral lagrangian by construction contains all terms compatible with the symmetries
  - Only a finite number of renormalised LECs need to be determined (pheno, lattice) for ChPT to make predictions.

Outline Part 4

- Loose ends (cont.)
  - Anomalies
  - Transformation law of the pion fields
- Some selected applications for Lattice QCD
  - Finite volume corrections
  - Non-zero lattice spacing corrections: Wilson fermions
  - Multi-pion excited state contribution

Anomalies

Anomalies

So far: $\mathcal{L}_{\mathrm{ChPT}} = \mathcal{L}_2 + \mathcal{L}_4 + \cdots$ based on local chiral invariance

Observation: all terms involve an even power of pion fields

➡ invariance under inner parity $\pi(x) \to -\pi(x)$

➡ experimentally observed processes involving an odd number of pion fields are not allowed/described

e.g. $K^+K^- \longrightarrow \pi^+\pi^-\pi^0$ $\pi^0 \longrightarrow \gamma\gamma$

Something is missing: anomalous Ward identities !

- Local chiral transformations are anomalous in the regularized theory
  There are additional anomalous contributions in the chiral Ward identities !
- In the path integral quantization anomalies stem from the non-invariance of the fermion measure in the fermionic path integral

Anomalies

Consider infinitesimal local transformations

$$R(x) = 1 + i\omega_R^a(x)T^a$$

$$L(x) = 1 + i\omega_L^a(x)T^a$$

$$\mathcal{D}[F'] = \mathcal{D}[F](1 + i\delta Z)$$

$$\delta Z = -\int d^4x\, \mathrm{tr}\Big(\omega_A(x)\Omega(x)\Big) \qquad \omega_A = \frac{1}{2}(\omega_R - \omega_L)$$

$\delta Z = \delta Z[v, a]$ independent of gluons, quarks, quark masses !

$$\Omega[v(x), a(x)] = \frac{N_c}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\big(v_{\mu\nu}(x)v_{\rho\sigma}(x) + \ldots$$

N_c : number of colors

field strength tensor $v_{\mu\nu} = \partial_\mu v_\nu - \ldots$

Important here: δZ is explicitly known !

Anomalies

$$\begin{aligned} W[v', a'] &= \int \mathcal{D}[F'] \, e^{iS[F', v', a']} \\ &= \int \mathcal{D}[F] (1 + i\delta Z[v, a]) \, e^{iS[F, v, a]} \\ &= \int \mathcal{D}[F] e^{iS[F, v, a] + i\delta Z[v, a]} \end{aligned}$$

in short: $$S_{\text{eff}}[F', v', a'] = S_{\text{eff}}[F, v, a] + \delta Z[v, a]$$

recall Lect 2, slide 11: $$W[v', a'] = W[v, a] + \delta W[v, a]$$

➥ $$\delta W[v, a] = \delta \mathcal{Z}[v, a] \qquad \delta \mathcal{Z}[v, a] \equiv \int \mathcal{D}[F] \delta Z[v, a] e^{iS[F, v, a]}$$

- Contains the WIs with the correct anomalous contributions
- $\delta \mathscr{Z}$ contributes to (some) 3,4,5 point functions involving the vector and axial vector currents

Anomalies

ChPT has to reproduce this: *Wess-Zumino-Witten (WZW) term* $Z[U, U^\dagger, v, a]_{\mathrm{WZW}}$

➥ $$Z[U', U'^\dagger, v', a']_{\mathrm{WZW}} = Z[U, U^\dagger, v, a]_{\mathrm{WZW}} + \delta Z[v, a]$$

same as in underlying QCD

Expand in pion fields*:

$$Z[U, U^\dagger, v, a]_{\mathrm{WZW}} = \int d^4x \frac{2}{15} \frac{N_c}{\pi^2 f^5} \epsilon^{\mu\nu\rho\sigma} \mathrm{tr}\big(\pi \partial_\mu \pi \partial_\nu \pi \partial_\rho \pi \partial_\sigma \pi\big) + \ldots$$

5 π-fields, contribution to

$K^+ K^- \longrightarrow \pi^+ \pi^- \pi^0$

If $a(x) = 0,\ v(x) = -eQA_\mu(x)$

$$\ldots + \int d^4x \frac{N_c}{3} \frac{e^2}{32\pi^2 f} \epsilon^{\mu\nu\rho\sigma} \partial_\mu \pi^0 \partial_\nu A_\rho A_\sigma + \ldots$$

contribution to

$\pi^0 \longrightarrow \gamma\gamma$

*Full expression e.g. in
J. Bijnens et. al, hep-ph/9411232

Anomalies

- Comment: Anomalies break the chiral WIs in a well-defined way
  ➥ ChPT is set up to reproduce this breaking
- WZW term has no contribution to 2pt functions
- WZW is $\mathrm{O}(p^4)$

➥
$$\mathcal{L}_{\mathrm{ChPT}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_{\mathrm{WZW}} + \mathcal{L}_6 + \mathcal{L}_8 + \cdots$$

- All $\mathcal{L}_{\mathrm{2n}}$ in here are *invariant under local chiral transformations*

Transformation law of the pion fields

Transformation law for the pion field

Recall Lecture 1, slide 44:

Question: How do the pion fields transform under G ?

Answer: There exists the 1-to-1 map

$$\pi^a(x) \longrightarrow U(x) = \exp\left[\frac{2i}{\tilde{f}}\pi^a(x)T^a\right] \qquad \dim[\tilde{f}] = 1$$

that maps the pion fields onto the *coset space* $G/H \simeq SU(3)$

$U(x)$ transforms extremely simple (linearly) under $R, L \in G$

$$U(x) \xrightarrow{G} R\,U(x)\,L^\dagger$$

Transformation law for the pion field

Consider Lie groups G and H ⊂ G

SSB ➞ n = dim G - dim H NGBs: π^a, a = 1,…, n; $\pi = (\pi^1, \ldots, \pi^n)$

Applying a transformation $g \in$ G

$$\pi \xrightarrow{g} \pi' = \varphi(g, \pi)$$

Mapping φ: G x M ⟶ M   M: set of pion fields

Properties of φ:

1. $\varphi(e, \pi) = \pi$
2. $\varphi(g_1 g_2, \pi) = \varphi(g_1, \varphi(g_2, \pi))$

First consider $\varphi(g, 0)$ $\pi = 0$ is the "ground state configuration"

Using 1. & 2. we can show:

a) Elements h with $\varphi(h, 0) = 0$ form a subgroup H ⊂ G

e.g. $\varphi(h_1 h_2, 0) = \varphi(h_1, \varphi(h_2, 0)) = \varphi(h_1, 0) = 0$

Transformation law for the pion field

a) implies that we can decompose the group in cosets, coset space = G/H

g_1, g_2 are in the same coset if $g_1 = g_2\, h$, $h \in$ H

b) the function $\varphi(g,0)$ is a function on the coset space G/H:

$$\varphi(gh,0) = \varphi(g,\varphi(h,0)) = \varphi(g,0) \qquad \forall\, g \in \mathrm{G} \qquad \forall\, h \in \mathrm{H}$$

⇒ *the map* $\varphi_0(g) \equiv \varphi(g,0)$ *defines a map from the coset space into the set of* π *fields*

$$\varphi_0 : \ \mathrm{G/H} \longrightarrow \mathrm{M}$$

c) The map φ_0 is invertible: Suppose $\varphi_0(g_1) = \varphi_0(g_2)$

$$\Rightarrow \quad 0 = \varphi(g_1^{-1}g_1,0) = \varphi_0(g_1^{-1},\varphi(g_1,0)) = \varphi_0(g_1^{-1},\varphi(g_2,0)) = \varphi(g_1^{-1}g_2,0)$$

⇒ $g_1^{-1}g_2 \in$ H $\qquad g_2 = g_1 h \qquad$ g_1, g_2 are in the same coset

⇒ *the map* $\varphi_0(g)$ *defines a 1-1-map*

Transformation law for the pion field

a), b), c) $\Rightarrow$ *the map $\varphi_0(g)$ defines a 1-1-map from the coset space into the set of π fields*

- to each π there exist a unique f in G/H
- the pion fields are the "coordinates" of the manifold G/H

Transformation of the pion fields: Consider $\pi = \varphi_0(f) = \varphi(f, 0)$

Act with $g \in$ G $\quad \pi \xrightarrow{g} \pi' = \varphi(g, \pi)$

$$= \varphi(g, \varphi(f, 0)) = \varphi(gf, 0) = \varphi(f', 0)$$

To get π' we "simply" need the coset of $g\,f = f'$

$$\begin{tikzcd} \pi \arrow[r, "g"] \arrow[d, leftrightarrow, "\varphi_0"'] & \pi' \arrow[d, leftrightarrow, "\varphi_0"] \\ f \arrow[r, "g"] & f' \end{tikzcd}$$

Transformation law for the pion field

Be specific for QCD: G = $SU(3)_R \otimes SU(3)_L$

H = $SU(3)_V$ V: R = L

$\Rightarrow$ G/H $\simeq$ SU(3) $U = \exp\left(\frac{2i}{f}\pi^a T^a\right)$

$$g = (R, L) \longrightarrow U = RL^{-1} \in G/H$$

$$\tilde{g} = gh = (RV, LV) \longrightarrow \tilde{U} = RV(LV)^{-1} = RVV^{-1}L^{-1} = U$$

Trafo of U under some $\bar{g}$

$$g' = \bar{g}g = (\bar{R}R, \bar{L}L) \quad \Rightarrow \quad U' = \bar{R}U\bar{L}^{-1}$$

linear transformation law

The transformation law of the pion fields is non-linear and more complicated

Finite volume ChPT

something before ...

Lattice QCD is formulated on euclidean space time

➥ continuum limit yields euclidean QCD

➥ corresponding low-energy effective theory is euclidean ChPT

➥ $$\mathcal{L}_2^{\rm eucl} = \frac{f^2}{4}{\rm tr}[\partial_\mu U \partial_\mu U^\dagger] - \frac{f^2 B}{2}{\rm tr}[M(U+U^\dagger]$$ cp. with Lect 2, slide 23

Modifications compared to Minkowski spacetime

- O(4) invariance
- opposite sign of mass term
- Pion propagator:

$$G_E(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2+M_\pi^2}e^{ip(x-y)}$$

Finite volume ChPT

Consider QCD on a torus with $V = L^4$ and periodic/antiperiodic boundary conditions

➥ What is the effect of the finite volume on correlation functions and observables?

Physical intuition (start from large or infinite volume and make it smaller)

- Lightest particles "*feel*" the FV first: $l_{\text{corr}} = 1/\,\mathrm{M}_{\text{particle}}$
  ➥ QCD: Dominant FV effect due to the pions (less from the kaons)

- FV effect is expected to be small if $l_{\text{corr}} \,/\, L$ is small , i.e. if $M_\pi L$ large
  ➥ Relevant measure is expected to be $M_\pi L$

Questions:
- How big are the FV corrections for a given $M_\pi L$?
- How rapidly is the infinite volume limit approached ?

Because the pions cause the dominant FV corrections
ChPT can answer these questions

Finite volume ChPT

Note: A 4-dimensional torus breaks Lorentz symmetry

Key results:

- The chiral Lagrangian in FV is the same as in infinite volume
  the LECs do not depend on the FV

J. Gasser, H. Leutwyler;
Nucl. Phys B 307 (1999) 763

Sketch of the argument:

Consider *finite temperature* QFT: $L_4 = \dfrac{1}{T}$ + boundary conditions for the fields

but the same Lagrangian with T independent parameters

For spatial finite volume invoke hypercubic symmetry: same Lagrangian with L independent parameters

$\Rightarrow$ *Finite volume effects stem from the pion propagator only*

FV pion propagator

The periodic boundary conditions imply

$$G_L(x-y) = G_L(x-y+n_\mu L_\mu) \qquad\qquad n_\mu \in \mathbb{Z}^4$$

➥ solution $$G_L(x-y) = \sum_{n_\mu} G_\infty(x-y+n_\mu L_\mu)$$

Fourier transform: Momenta are discrete $$p_\mu = \frac{2\pi}{L} n_\mu$$

$$G_L(x-y) = \frac{1}{L_1 L_2 L_3 L_4} \sum_p \frac{e^{ip(x-y)}}{p^2 + M_\pi^2}$$

cp. with slide 17

Feynman rules for the vertices are the same as in infinite volume

Example: FV correction to the pion mass

Simplifications
- SU(2) ChPT, $m_{\mathrm{u}} = m_{\mathrm{d}} = m$, tree level pion mass: $M_0^2 = 2Bm$
- $L_4 = \infty$, $L_1 = L_2 = L_3 = L$ (finite *spatial* volume)

$$
\begin{aligned}
G_L(x-y) &= \sum_{n_k} G_\infty(x-y+n_k L_k) \\
&= G_\infty(x-y) + \underbrace{\sum_{n_k\neq 0} G_\infty(x-y+n_k L_k)}_{\delta_{\mathrm{FV}}(x-y)}
\end{aligned}
$$

This expression enters the loop diagrams for the self energy for $x-y=0$
(cp. Lecture 2, slides 8/9)

$$
G_\infty(0) \xrightarrow{\text{D dim}} \frac{M_0^2}{16\pi^2}\left(-\frac{2}{\epsilon} + \ln M_0^2 + \text{finite}\right)
$$

For δ_{FV} use the finite expression $\quad G_\infty(z) = \dfrac{1}{4\pi^2}\dfrac{M_0}{\sqrt{z^2}} K_1\left(M_0\sqrt{z^2}\right)$

modified Bessel function

Example: FV correction to the pion mass

We obtain for $z = (\vec{n}L, 0)$ *"pions wrapping around the torus"*

$n = \vec{n}^2$ (!)

$$\delta_{\rm FV}(0) = \frac{M_0^2}{16\pi^2} \sum_{n\neq 0} 4m(n) \frac{K_1(\sqrt{n}M_0L)}{\sqrt{n}M_0L}$$

$m(n)$: multiplicities

$m(1)=6,\ m(2)=12,\ m(3)=8, \ldots$

Putting both together leads to the simple replacement rule

$$\frac{M_0^2}{16\pi^2}\ln\frac{M_0^2}{\Lambda_3^2} \longrightarrow \frac{M_0^2}{16\pi^2}\ln\frac{M_0^2}{\Lambda_3^2} + \delta_{\rm FV}(0)$$

$$\Rightarrow \quad M_\pi^2 = M_0^2\left(1 + \frac{M_0^2}{32\pi^2 f^2}\ln\frac{M_0^2}{\Lambda_3^2} + \frac{M_0^2}{32\pi^2 f^2}\sum_{n\neq 0} 4m(n)\frac{K_1(\sqrt{n}M_0L)}{\sqrt{n}M_0L}\right)$$

$$= M_{\pi,\infty}^2\left(1 + \frac{M_\pi^2}{32\pi^2 f^2}\sum_{n\neq 0} 4m(n)\frac{K_1(\sqrt{n}M_\pi L)}{\sqrt{n}M_\pi L}\right)$$

Here: drop higher order corrections $(1 + \ln + \delta_{\rm FV}) \approx (1 + \ln)(1 + \delta_{\rm FV})$

Example: FV correction to the decay constant

Analogously, we obtain the FV correction for the pion decay constant:

$$f_{\pi,L} = f\Big(1 - \frac{M_0^2}{16\pi^2 f^2}\ln\frac{M_0^2}{\Lambda_4^2} - \frac{M_0^2}{16\pi^2 f^2}\sum_{n\neq 0} 4m(n)\frac{K_1(\sqrt{n}M_0 L)}{\sqrt{n}M_0 L}\Big)$$

$$= f_{\pi,\infty}\Big(1 - \frac{M_\pi^2}{16\pi^2 f^2}\sum_{n\neq 0} 4m(n)\frac{K_1(\sqrt{n}M_\pi L)}{\sqrt{n}M_\pi L}\Big)$$

Size of the FV corrections

There exist various representations for Bessel function K_1, e.g.

$$K_1(z) = \frac{z}{4}\int_0^\infty dt\; e^{-\left(t+\frac{z^2}{4t}\right)}\,\frac{1}{t^2}$$

Perform a saddle point expansion for large z

$$K_1(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z} \qquad \text{for} \qquad z \longrightarrow \infty$$

Exercise: Show this

$$\Rightarrow \qquad f_{\pi,L} = f_{\pi,\infty}\left[1 - \frac{M_\pi^2}{16\pi^2 f^2}24\sqrt{\frac{\pi}{2}}\frac{e^{-M_\pi L}}{(M_\pi L)^{3/2}} + \mathrm{O}\left(e^{-\sqrt{2}M_\pi L}\right)\right]$$

- FV corrections are *exponentially* suppressed
- As anticipated: The relevant quantity is the dimless number $M_\pi L$
- As anticipated: FV corrections of kaons and etas are even smaller than for pions

Range of applicability

- The results derived here apply only for $M_\pi L \gg 1$
  so called *p-regime*
- Do not hold for smaller values with $M_\pi L \ll 1$
  so-called *ε-regime*

zero-mode contribution

$$G_L(x-y) = \frac{1}{L^4}\sum_p \frac{e^{ip(x-y)}}{p^2+M_\pi^2} = \frac{1}{M_\pi^2 L^4} + \sum_{p\neq 0}\cdots$$

Reason: The zero-mode contribution becomes large

➥ reordering of the perturbative expansion becomes necessary

For details see Maarten Golterman, arXiv: 0912.4042

Final comments on FV ChPT

So far: periodic boundary conditions

- Twisted boundary conditions (periodic up to global symmetry transformations)
  ➥ Similar results   G.Colangelo and A.Vaghi, 2016

- Dirichlet or open boundary conditions in time
  ➥ ???
  More complicated because of

  - "real" boundaries for $x_0 = 0$ and $x_0 = T$

$$\mathcal{L} = \mathcal{L}_{\text{inf Vol}} + \mathcal{L}_{x_0=0} + \mathcal{L}_{x_0=T}$$

  - the boundary terms are expected to break chiral symmetry
    ➥ new terms with new unknown LECs appear in the chiral Lagrangian
    (not worked out so far …)

Non-zero lattice spacing corrections

Wilson ChPT

ChPT at non-zero lattice spacing

- Very often in practice: Lattice data at non-zero lattice spacing
  Desired: Apply ChPT to these results

- Not straightforward because

  - the lattice breaks Lorentz resp. O(4) invariance

  - most lattice fermions break chiral symmetry,
    e.g. Wilson and staggered fermions

  These breakings lead to modifications in the continuum ChPT results!

- In the following: Wilson fermions (simplest case ...)

ChPT at non-zero lattice spacing

- Constructing ChPT for non-zero lattice spacing:
  Strategy: Two-step matching to effective theories:

Energy

$$\mathcal{L}_{\mathrm{Lattice-QCD}}$$

$$\mathcal{L}_{\mathrm{Sym}} = \mathcal{L}_{\mathrm{QCD}} + a\mathcal{L}_1 + a^2\mathcal{L}_2 + \ldots$$

Symanzik effective theory

$$\mathcal{L}_{\mathrm{chiral}} = \mathcal{L}_{\mathrm{ChPT}} + a\tilde{\mathcal{L}}_1 + a^2\tilde{\mathcal{L}}_2 + \ldots$$

Inherited: Details depend on the Lattice Dirac operator

- "Lattice ChPT" is an expansion in powers of small pion momenta, pion mass and a small lattice spacing a

e.g.

LO: $\mathcal{O}(p^2, m, a)$

NLO: $\mathcal{O}(p^4, p^2m, m^2, p^2a, ma, a^2)$

Not universal, different countings exist !

- Additional LECs associated to the "lattice spacing terms" appear

Example: Wilson fermions

Symanzik effective theory through $\mathrm{O}(a)$:

$$\mathcal{L}_{\rm Sym} = \mathcal{L}_{\rm QCD} + a\, c\, \overline{q} i\sigma_{\mu\nu} G_{\mu\nu} q + \mathrm{O}(a^2)$$

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

- One term (Pauli term)
  ("anomalous color magnetic moment, strength c , flavor independent)
- Breaking of chiral symmetry also allows a term $\frac{1}{a}\overline{q}q$
  ➥ included in the quark mass term in $\mathcal{L}_{\rm QCD}$

Pauli term breaks chiral symmetry like a mass term

$$\mathcal{L}_{\rm Pauli} = \overline{q}_R A\, i\sigma_{\mu\nu} G_{\mu\nu} q_L + \overline{q}_L A^\dagger i\sigma_{\mu\nu} G_{\mu\nu} q_R \qquad A = ac$$

➥ invariant if

$$A \xrightarrow{R\,,\,L} RAL^\dagger$$

$$A^\dagger \xrightarrow{R\,,\,L} LA^\dagger R^\dagger$$

$$A \xrightarrow{P} A^\dagger$$

A: spurion field
analogous to quark mass matrix
cp. with Lecture 2, slide 20

"physical value": $A = A^\dagger = ac$

Example: Wilson fermions

Additional invariant with the spurion field A : $\mathrm{Tr}[AU^\dagger + A^\dagger U]$

$$\Rightarrow \qquad \mathcal{L}_a[a] = -\frac{f^2 W_0\, c\, a}{2} \mathrm{tr}[U + U^\dagger]$$

- Leading term with one power of a
  recall: euclidean space-time $\rightarrow$ sign

- Form of a flavor diagonal mass term
  same breaking in QCD leads to the same terms in ChPT

- W_0: new LEC associated with the non-zero lattice spacing
  analogous to B associated with the non-zero mass term
  note:

  dim W_0 = 3, dim B = 1 $\quad\Rightarrow\quad$ dim $W_0 a$ = dim Bm_q = 2

- Convention: $W_0\, c \rightarrow W_0$ since both are unknown coefficients

Example: Wilson fermions

Higher order terms: Obtain these from the Lagrangian $\mathcal{L}_4$ by replacing

$$\chi = 2BM_q \longrightarrow 2W_0 a = \rho$$

$$L_k \longrightarrow W_k, W_k'$$

$\Rightarrow$ $$\mathcal{L}_a[p^2 a, m_q a] = W_4 \rho\, \mathrm{tr}[\partial_\mu U \partial_\mu U^\dagger]\, \mathrm{tr}[U + U^\dagger] \text{ + } W_5 \text{ term}$$
$$-W_6 \rho\, \mathrm{tr}[\chi(U + U^\dagger)]\, \mathrm{tr}[U + U^\dagger] \text{ + } W_7, W_8 \text{ terms}$$

$$\mathcal{L}_{a^2}[a^2] = -W_6' \rho^2 \left(\mathrm{tr}[U + U^\dagger]\right)^2 \text{+ } W_7', W_8' \text{ terms}$$

We find *8 additional terms* in the chiral Lagrangian !

Example: Wilson fermions

Comment on the power counting:

We count $p^2 \sim m_q$ because of $p^2 = M^2_{\rm tree} \propto m_q$

An analogous argument does not hold for the a contribution
The lattice spacing a can be changed independently of p^2 and m_q

Different countings exist depending on the relative size of m_q and a

$m_q \sim a$ *GSM regime* *generically small quark masses* → LO: $\mathrm{O}(p^2, m_q, a)$
NLO: $\mathrm{O}(p^4, p^2 m_q, {m_q}^2, p^2 a, m_q a, a^2)$

$m_q \sim a^2$ *LCE regime* *large cut-off effects* → not here …

Example: Wilson fermions at O(a)

Symanzik effective theory through O(a^2):

Sheikholeslami, Wohlert 1985

MANY more terms!

18 fermion operators (dim 6)
+ gluonic ones

Some examples:

- quark bilinears $O_1^{(6)} = \bar{q}(\gamma_\mu D_\mu)^3 q$ $O_5^{(6)} = \bar{q} M_q D_\mu D_\mu q$ + 6 more terms
- 4-quark operators $O_{16}^{(6)} = (\bar{q}\gamma_\mu T^a_{\rm color} q)^2$ $O_9^{(6)} = (\bar{q}q)^2$ + 8 more terms

chiral symmetry preserving

chiral symmetry breaking

All these operators need to be mapped to ChPT …

Example: Wilson fermions at $\mathrm{O}(a)$

Chiral symmetry preserving operators:

do not change the symmetry properties of continuum QCD

➥ map to the same continuum chiral Lagrangian

but: the LECs differ and depend on a^2

$$f \longrightarrow f(a^2) = f + f'a^2 + \ldots \quad \Rightarrow \quad \text{NNLO terms of } \mathrm{O}(p^2a^2)$$

$$L_k \longrightarrow L_k(a^2) = L_k + L_k'a^2 + \ldots \quad \Rightarrow \quad \text{N}^3\text{LO terms of } \mathrm{O}(p^4a^2)$$

Can be ignored if working at NLO only !

Example: Wilson fermions at O(a)

Chiral symmetry breaking operators:

Introduce spurion fields for each term and use to construct invariants in ChPT

➥ additional terms in the chiral Lagrangian at O(a^2)

but: no new terms, only the same ones we already found using $A^2,\ AA^\dagger,\ \ldots$

OB, Rupak, Shoresh 2004

Consequence: Effectively only the LECs W_6', W_7', W_8' change

$$W_6'\rho^2\left(\mathrm{tr}[U+U^\dagger]\right)^2 \longrightarrow \quad W_6'\rho^2\left(\mathrm{tr}[U+U^\dagger]\right)^2 + \sum_j \tilde{W}_{6,j}'\rho^2\left(\mathrm{tr}[U+U^\dagger]\right)^2$$
$$= (W_6' + \sum_j \tilde{W}_{6,j}')\,\rho^2\left(\mathrm{tr}[U+U^\dagger]\right)^2$$

Upshot: The 18 operators at O(a^2) in the Symanzik effective action do not qualitatively change the chiral Lagrangian up to NLO

Example: $\pi\pi$ scattering in WChPT

Results for scattering amplitude T for $\quad \pi^+(p_1)\ \pi^+(p_2) \longrightarrow \pi^+(p_1')\ \pi^+(p_2')$

SU(2) ChPT, $m_u = m_d = m$, GSM regime

➥ Consider threshold value, i.e. for $\quad s = 4M_\pi^2, t = u = 0$

LO: $\quad T|_{\rm thr} = -\frac{2M_0^2}{f^2}$ continuum result, no $\mathrm{O}(a)$ correction ! cp. Lecture 2, slide 28

The $\mathrm{O}(a)$ term is contained in the pion mass

NLO: $$T|_{\rm thr} = -\frac{2M_0^2}{f^2}\left(\left[1 - \frac{4}{3}\frac{M_0^2}{(4\pi f)^2}\ln\frac{M_0^2}{\Lambda_1^2}\right] - k_1\frac{\rho}{f^2}\right) + 32(2W_6' + W_8')\frac{\rho^2}{f^2}$$

$k_1 = k_1(W_4, \ldots, W_8) \qquad \rho = 2W_0 a$

- We recover the continuum results for $a \to 0$
- Note: Chiral and continuum limit do not commute !
  You enter the LCE regime first …

More applications

- More applications
  - twisted mass QCD $\longrightarrow$ twisted mass ChPT
  - staggered QCD $\longrightarrow$ staggered ChPT
  - mixed action QCD $\longrightarrow$ mixed action ChPT
    different lattice fermions
    for sea and valence quarks

- Introduction including many references to original papers:

  M. Golterman, *Applications of chiral perturbation theory to lattice QCD*
  Les Houches lecture notes, arXiv:0912.4042 [hep-lat]

Lorentz or O(4) symmetry breaking

O(4) breaking

Symanzik effective theory through $\mathrm{O}(a^2)$:

Sheikholeslami, Wohlert 1985

$$
\begin{aligned}
O_4^{(6)} &= \overline{q}\gamma_\mu D_\mu D_\mu D_\mu q \\
&= \overline{q}_R\gamma_\mu D_\mu D_\mu D_\mu q_R + \overline{q}_L\gamma_\mu D_\mu D_\mu D_\mu q_L
\end{aligned}
$$

- breaks O(4) symmetry
- preserves chiral symmetries

  also present for chiral lattice fermions (Ginsparg-Wilson, Domain-wall)

Exercise: Map this term to ChPT

Multi-pion state contamination

Pion pole dominance revisited

We saw (Lecture 2, slide 35) $$\langle 0|T A^1_\mu(x) A^1_\nu(y)|0\rangle = f^2 \frac{p_\mu p_\nu}{p^2+M_\pi^2} \quad + \quad \ldots$$

$$\langle 0|T A^4_\mu(x) A^4_\nu(y)|0\rangle = f^2 \frac{p_\mu p_\nu}{p^2+M_K^2} \quad + \quad \ldots$$

- The pion pole is the dominant part (lowest lying pole) reproduced in ChPT
- $\ldots$ stands for poles of non-GB particles
- There exist also multi-pion contributions e.g. 3-particle states: $\pi\pi\pi$ or $K\pi\pi$

  These are contained in ChPT !

Example: pion correlator

$\boxed{\pi^0}$ $\qquad I^G(J^{PC}) = 1^-(0^{-+})$

We have omitted some results that have been superseded by later experiments. The omitted results may be found in our 1988 edition Physics Letters **B204** 1 (1988).

π^0 MASS

VALUE (MeV)	*DOCUMENT ID*
134.9768±0.0005 OUR FIT	Error includes scale factor of 1.1.

$\boxed{\pi(1300)}$ $\qquad I^G(J^{PC}) = 1^-(0^{-+})$

Same quantum numbers

$\pi(1300)$ MASS

VALUE (MeV)	*EVTS*	*DOCUMENT ID*	*TECN*	*COMMENT*
1300±100 OUR ESTIMATE				

3π state with all pions at rest: $E_{3\pi} \approx 420\ \mathrm{MeV} \ll 1300\ \mathrm{MeV}$

$\pi\pi\pi$ states may have a larger impact than $\pi(1300)$

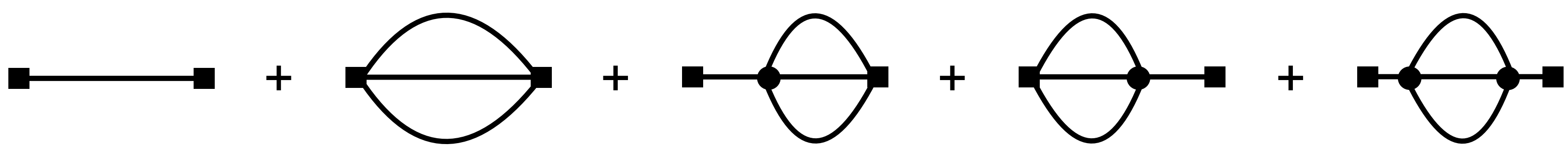
Multi-pion states in ChPT

Consider QCD in a finite spatial volume $V = L^3$ $\Rightarrow$ discrete spatial momenta
➥ isolated poles in the spectral decomposition

$$C_{A^4}(t) = \int d^3x \langle A_0^4(\vec{x},t) A_0^4(\vec{0},0)\rangle$$

$$= c_0\, e^{-M_K t} \;+\; c_1\, e^{-(M_K+2M_\pi)t} \;+\; \ldots$$

Feynman diagrams:



$$c_0 = \frac{f^2 M_K}{2} \qquad\qquad \frac{c_1}{c_0} = \frac{3}{128(fL)^4(M_\pi L)^2}\frac{M_K^2}{(M_K+M_\pi)^2}$$

Well defined (finite) expression

Multi-pion states in ChPT

- Relevance in practice: Excited-state contamination in calculating M_K and f_K typically extracted from fits of a single exponential (c_0 part) to lattice data

- Not really an issue for the pion and kaon correlation functions

  - Euclidean time t can be chosen large enough such that the excited state contribution is negligible, e.g. $K\pi\pi$ contribution is $O(10^{-4})$ for $t \approx 1$ fm

- However: Much larger contribution in *nucleon obeservables* like nucleon form factors
  ➥ dominant excited state contribution are 2-particle $N\pi$ states
  For a recent review see OB, arXiv:1708.00380 [hep-lat]

References

Most of what was covered here can be found in:

- J. Bijnens, G. Ecker and J. Gasser: ***Chiral Perturbation Theory***, hep-ph/9411232
  ➠ Anomalies and the WZW term
- Heinrich Leutwyler: ***Principles of chiral perturbation Theory***, hep-ph/9406283
  ➠ Anomalies, Transformation of pion fields
- Stefan Scherer, Matthias Schindler, ***A Primer for Chiral Perturbation Theory***, Lecture Notes in Physics 830, Springer
  ➠ Anomalies, Transformation of pion fields
- Maarten Golterman: ***Applications of chiral perturbation theory to lattice QCD***, arXiv:0912.4042 [hep-lat]
  ➠ Finite volume ChPT, Wilson ChPT

Exercises

For possible exercises see

- slide 24
- slide 40