Non-perturbative Renormalization and Improvement of Lattice QCD

Lecture 3

Stefan Sint

Trinity College Dublin

EuroPLEx online school

Dublin, 21 October 2020

Contents

Step Scaling Functions

- Given $\bar{g}(L)$ and $\bar{m}(L)$, the aim is to construct the Step Scaling Functions $\sigma(u)$ and $\sigma_{\mathrm{P}}(u)$:

$$\begin{aligned}\sigma(u) &= \bar{g}^2(2L)\big|_{u=\bar{g}^2(L)},\\ \sigma_{\mathrm{P}}(u) &= \lim_{a\to 0} \frac{Z_{\mathrm{P}}(g_0, 2L/a)}{Z_{\mathrm{P}}(g_0, L/a)}\bigg|_{u=\bar{g}^2(L)}\end{aligned}$$

- These are related to the usual RG functions:

$$\int_{\sqrt{\sigma(u)}}^{\sqrt{u}} \frac{\mathrm{d}g}{\beta(g)} = \ln 2 \qquad \sigma_{\mathrm{P}}(u) = \exp \int_{\sqrt{\sigma(u)}}^{\sqrt{u}} \frac{\tau(g)}{\beta(g)} \mathrm{d}g$$

- One thus considers a change of scale by a finite factor $s = 2$; RG functions β and τ tell us what happens for infinitesimal scale changes.

Lattice approximants $\Sigma(u, a/L)$ for $\sigma(u)$

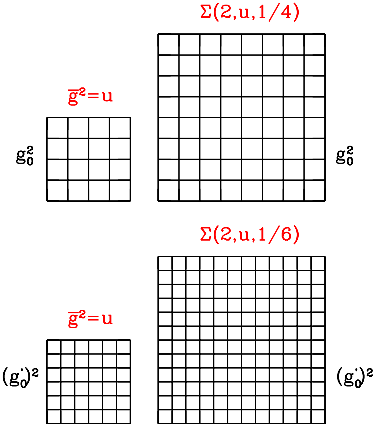
- choose g_0 and $L/a = 4$, measure $\bar{g}^2(L) = u$ (this sets the value of u)
- double the lattice and measure

$$\Sigma(u, 1/4) = \bar{g}^2(2L)$$

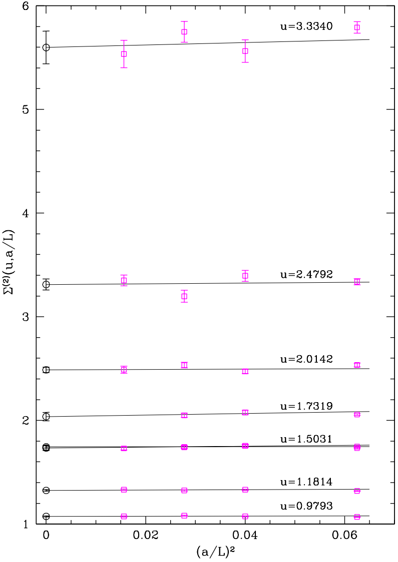
- now choose $L/a = 6$ and tune g_0' such that $\bar{g}^2(L) = u$ is satisfied
- double the lattice and measure

$$\Sigma(u, 1/6) = \bar{g}^2(2L)$$

- and so on ...

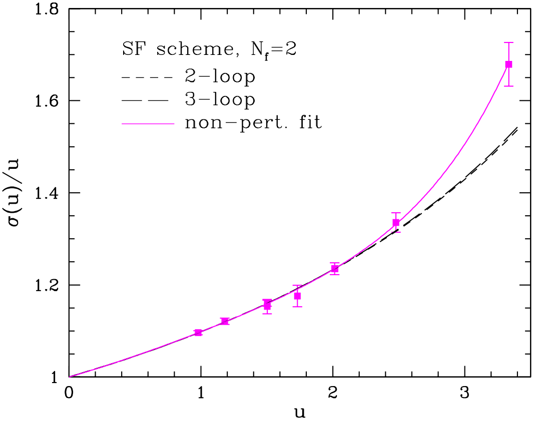


Continuum extrapolation of the SSF [ALPHA '05]



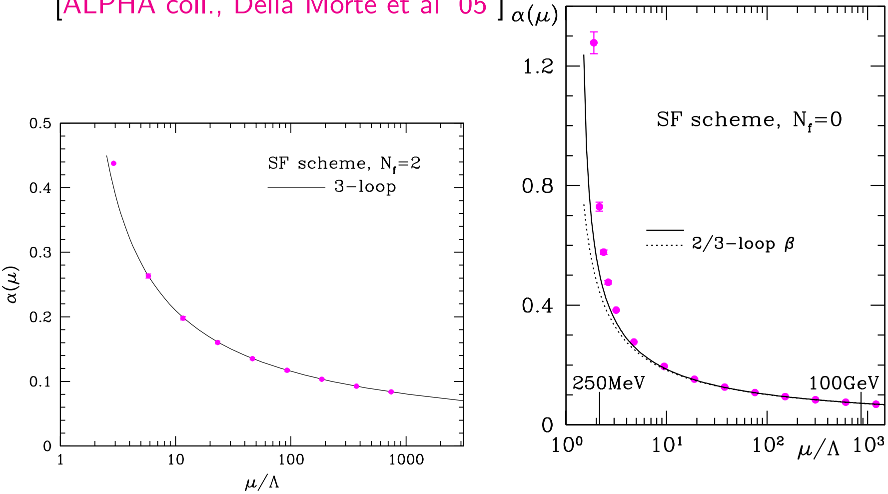
The SSF in the continuum limit

[ALPHA coll., Della Morte et al '05]



The running of the SF coupling

[ALPHA coll., Della Morte et al '05]



Determination of the Λ-parameter

- The formula

$$\Lambda = \mu\left(b_0\bar{g}^2\right)^{-b_1/2b_0^2} \exp\left\{-\frac{1}{2b_0\bar{g}^2}\right\} \times \exp\left\{-\int_0^{\bar{g}} \mathrm{d}x \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\}$$

holds for any value of μ. We may use it at L_{min} to obtain

$$\Lambda L_{\mathrm{min}} = f\left(\bar{g}(L_{\mathrm{min}})\right)$$

- The function $f(g)$ can be evaluated at $g = \bar{g}(L_{\mathrm{min}})$ since this is deep in the perturbative region. The integral in the exponent

$$\int_0^{\bar{g}} \mathrm{d}x \left[\frac{b_2 b_0 - b_1^2}{b_0^3} x + \mathrm{O}(x^3)\right] = \frac{b_2 b_0 - b_1^2}{2b_0^3}\bar{g}^2 + \mathrm{O}(\bar{g}^4)$$

may thus be evaluated using the β-function at 3-loop order.
- Since $L_{\mathrm{max}} = 2^n L_{\mathrm{min}}$ one knows $L_{\mathrm{max}}\Lambda$
- still need $F_\pi L_{\mathrm{max}}$

Matching to a low energy scale

Ideally one would like to compute e.g. $F_\pi\Lambda$, and take $F_\pi = 132\mathrm{MeV}$ from experiment

- What is required? The scale $L_{\max}$ is implicitly defined:

$$\bar{g}^2(L_{\max}) = 4.84 \qquad \Rightarrow \qquad (L_{\max}/a)(g_0)$$

  Setting $L_{\max}/a = 6, 8, 10, \ldots$ one then finds corresponding values of the bare coupling (at fixed g_0 some interpolation of $L_{\max}/a$ will be necessary instead)
- One must then be able to compute aF_π in a large volume simulation at the very same values of the bare coupling:

$$F_\pi\Lambda = \lim_{g_0\to 0}(L_{\max}/a)(g_0)(aF_\pi)(g_0)$$

- One thus needs a range of g_0 where both can be computed, aF_π and $\bar{g}(L_{\max})$
- Remark: intermediate results are often quoted in terms of Sommer's scale r_0, rather than F_π.

Results for QCD with $N_\mathrm{f} = 0$ and $N_\mathrm{f} = 2$ quark flavours

- The scale r_0 [R. Sommer '93] is obtained from the force $F(r)$ between static quark and antiquark separated by a distance r:

$$r_0^2 F(r_0) = 1.65$$

The r.h.s. was chosen so that phenomenological estimates from potential models yield $r_0 = 0.5\,\mathrm{fm}$.

- Recent result for $N_\mathrm{f} = 2$ ([ALPHA '12]): $F_K = 155\,\mathrm{MeV}$ implies $r_0 = 0.503(10)\,\mathrm{fm}$ (at physical pion mass!).
- Results for Λ using $r_0 = 0.5\,\mathrm{fm}$ [ALPHA '99-'12]

$$\begin{aligned} \Lambda^{(2)}_{\overline{\mathrm{MS}}} r_0 &= 0.789(52), & \Lambda^{(2)}_{\overline{\mathrm{MS}}} &= 310(20)\,\mathrm{MeV} \\ \Lambda^{(0)}_{\overline{\mathrm{MS}}} r_0 &= 0.602(48), & \Lambda^{(0)}_{\overline{\mathrm{MS}}} &= 238(19)\,\mathrm{MeV} \end{aligned}$$

The running quark mass

- Coupled evolution of the running mass and the coupling:

$$\begin{aligned}\overline{m}(2L) &= \sigma_m(u)\overline{m}(L), \qquad \sigma_m(u) = 1/\sigma_{\mathrm{P}} \\ \bar{g}^2(2L) &= \sigma(u)\end{aligned}$$

- Once the running coupling is known in a range $[u_0, u_n]$,

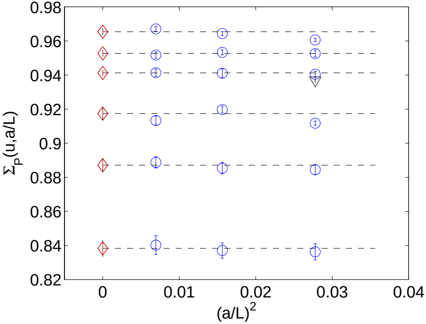
$$u_0 = \bar{g}^2(L_{\mathrm{min}}), \quad u_k = \bar{g}^2(2^k L_{\mathrm{min}}), k = 1, 2, \ldots, n$$

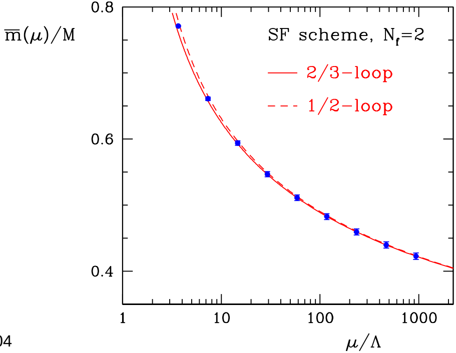
  determine $\sigma_m(u)$ for the same range of couplings: evolution of quark mass and coupling recursively

$$\overline{m}(2^k L_{\mathrm{min}})/\overline{m}(2^{k-1} L_{\mathrm{min}}) = \sigma_m(u_k), \qquad k = 1, 2, \ldots, n$$

- one obtains $\overline{m}(2L_{\mathrm{max}})/\overline{m}(L_{\mathrm{min}})$
- Extract $\overline{m}(L_{\mathrm{min}})/M$ using PT as for Λ-parameter

Running mass in the SF scheme [ALPHA '05]





Relation to bare quark masses

- In practice with Wilson type quarks, one avoids the additive renormalisation of the bare quark mass parameter by replacing it by a *measured* bare mass m_{PCAC} from the (bare) PCAC relation:

$$m_{\mathrm{PCAC}} \stackrel{\mathrm{def}}{=} \frac{\langle \partial_\mu A_\mu^a(x) O \rangle}{2\langle P^a(x) O \rangle}$$

- The running quark mass is then related to m_{PCAC}

$$\overline{m}(L) = \underbrace{Z_{\mathrm{P}}^{-1}(g_0, L/a) Z_{\mathrm{A}}(g_0)}_{\text{known factors}} \underbrace{m_{\mathrm{PCAC}}(g_0)}_{\text{measured}},$$

- Combine results,

$$M = Z_M(g_0) m_{\mathrm{PCAC}}(g_0)$$

and take the continuum limit $g_0 \to 0$.

Strange quark mass

$N_\mathrm{f} = 2$ result for the strange quark mass using this strategy ([ALPHA '12]):

$$M_s = 138(3)(1)\,\mathrm{MeV} \quad \Rightarrow \quad \overline{m}^{\overline{\mathrm{MS}}}(\mu = 2\,\mathrm{GeV}) = 102(3)(1)\,\mathrm{MeV}$$

- Quoted errors are statistical and systematic;
- <u>Note</u>: Except for quenching of the strange quark the ALL systematic errors have been addressed!

Summary step scaling

- The recursive finite volume technique has completely eliminated the problem with large scale differences.
- Physical results require a matching calculation at a low energy scale: it is crucial to have a range in bare couplings where both, the renormalisation conditions and the hadronic input can be computed
- State of the art [ALPHA '17]: QCD with $N_f = 3$ & technical refinements:
  - gradient flow coupling in finite volume scheme at low energies
  - Global fits to the step-scaling function
  - Improved control of cutoff effects
  - use of the flow time scale t_0 instead of r_0 as intermediate scale
- Many more applications in the literature: all quark bilinear operators, $\Delta F = 2$ 4-quark operators, HQET,..

Renormalized lattice QCD with Wilson quarks

The action $S = S_{\rm f} + S_{\rm g}$ is given by

$$S_{\rm f} = a^4 \sum_x \overline{\psi}(x)\,(D_W + m_0)\,\psi(x), \qquad S_{\rm g} = \frac{1}{g_0^2}\sum_{\mu,\nu} \mathrm{tr}\,\{1 - P_{\mu\nu}(x)\}$$

$$D_W = \frac{1}{2}\left\{\left(\nabla_\mu + \nabla_\mu^*\right)\gamma_\mu - a\nabla_\mu^*\nabla_\mu\right\}$$

- Symmetries: $\mathrm{U}(N_{\rm f})_{\rm V}$ (mass degenerate quarks), P, C, T and $O(4,\mathbb{Z})$

$\Rightarrow$ Renormalized parameters:

$$g_{\rm R}^2 = Z_g g_0^2, \qquad m_{\rm R} = Z_m\,(m_0 - m_{\rm cr})\,, \qquad am_{\rm cr} = am_{\rm cr}(g_0).$$

- In general: $Z = Z(g_0^2, a\mu, am_0)$;
- Quark mass independent renormalisation schemes: $Z = Z(g_0^2, a\mu)$
- Simple non-singlet composite fields, e.g. $P^a = \overline{\psi}\gamma_5\tau^a\psi$ renormalise multiplicatively, $P^a_{\rm R} = Z_{\rm P}(g_0^2, a\mu, am_0)P^a$

Approach to the continuum limit (1)

Suppose we have renormalised lattice QCD non-perturbatively, how is the continuuum limit approached?

Symanzik's effective continuum theory [Symanzik '79]:

- purpose: render the a-dependence of lattice correlation functions explicit. $\Rightarrow$ structural insight into the nature of cutoff effects
- at scales far below the cutoff a^{-1}, the lattice theory is effectively continuum like; the influence of cutoff effects is expanded in powers of a:

$$\begin{aligned} S_{\mathrm{eff}} &= S_0 + aS_1 + a^2 S_2 + \ldots, \qquad S_0 = S_{\mathrm{QCD}}^{\mathrm{cont}} \\ S_k &= \int \mathrm{d}^4 x \, \mathcal{L}_{\mathrm{k}}(x) \end{aligned}$$

$\mathcal{L}_{\mathrm{k}}(x)$: linear combination of fields

- with canonical dimension $4 + k$
- which share all the symmetries with the lattice action

Approach to the continuum limit (2)

A complete set of dimension 5 fields for $\mathcal{L}_1$ is given by:

$$\overline{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi, \qquad \overline{\psi}D_\mu D_\mu\psi, \qquad m\,\overline{\psi}\not{D}\psi, \quad m^2\overline{\psi}\psi, \qquad m\,\mathrm{tr}\,\{F_{\mu\nu}F_{\mu\nu}\}$$

The same procedure applies to composite fields:

$$\phi_{\rm eff}(x) = \phi_0 + a\phi_1 + a^2\phi_2 \ldots$$

for instance: $\phi(x) = P^a(x)$, basis for ϕ_1:

$$m\,\overline{\psi}\gamma_5\tfrac{1}{2}\tau^a\psi, \qquad \overline{\psi}\overleftarrow{\not{D}}\gamma_5\tfrac{1}{2}\tau^a\psi - \overline{\psi}\gamma_5\tfrac{1}{2}\tau^a\not{D}\psi$$

Consider renormalised, connected lattice n-point functions of a multiplicatively renormalisable field ϕ

$$G_n(x_1,\ldots,x_n) = Z_\phi^n\langle\phi(x_1)\cdots\phi(x_n)\rangle_{\rm con}$$

Approach to the continuum limit (3)

Effective field theory description:

$$\begin{aligned} G_n(x_1,\ldots,x_n) &= \langle \phi_0(x_1)\ldots\phi_0(x_n)\rangle_{\rm con} \\ &\quad + a\int \mathrm{d}^4y\ \langle \phi_0(x_1)\ldots\phi_0(x_n)\mathcal{L}_1(y)\rangle_{\rm con} \\ &\quad + a\sum_{k=1}^{n}\langle \phi_0(x_1)\ldots\phi_1(x_k)\ldots\phi_0(x_n)\rangle_{\rm con} + \mathrm{O}(a^2) \end{aligned}$$

- $\langle\cdots\rangle$ is defined w.r.t. continuum theory with S_0
- the a-dependence is now explicit, up to logarithms, which are hidden in the coefficients.
- In perturbation theory one expects at l-loop order:

$$P(a) \sim P(0) + \sum_{n=1}^{\infty}\sum_{k=1}^{l} c_{nk}\, a^n (\ln a)^k$$

where e.g. $P(a) = G_n$ at fixed arguments.

Approach to the continuum limit (4)

Conclusions from Symanzik's analysis:

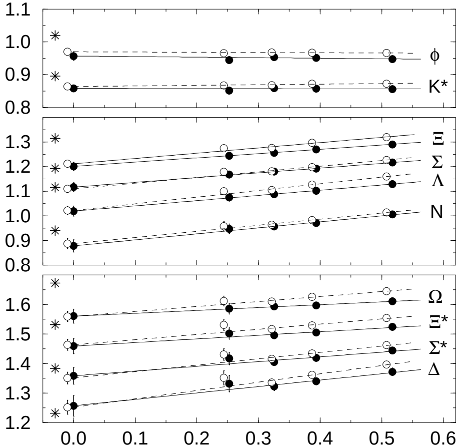
- Asymptotically, cutoff effects are powers in a, modified by logarithms;
- In contrast to Wilson quarks, only even powers of a are expected for
  - bosonic theories (e.g. pure gauge theories, scalar field theories)
  - fermionic theories which retain a remnant axial symmetry (overlap, Domain Wall Quarks, staggered quarks, Wilson quarks with a twisted mass term, etc.)

In QCD simulations a is typically varied by a factor 2

$\Rightarrow$ logarithms vary too slowly to be resolved; linear or quadratic fits (in a resp. a^2) are used in practice.

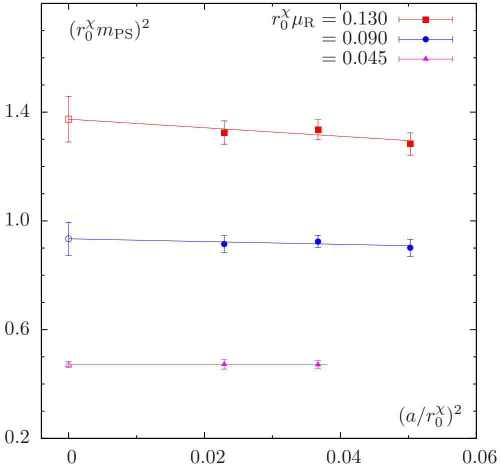
Example 1: quenched hadron spectrum

Linear continuum extrapolation of the quenched hadron spectrum; standard Wilson quarks with Wilson's plaquette action:[CP-PACS coll., Aoki et al. '02] $a = 0.05 - 0.1\,\mathrm{fm}$, experimental input: $m_K,\ m_\pi,\ m_\rho$



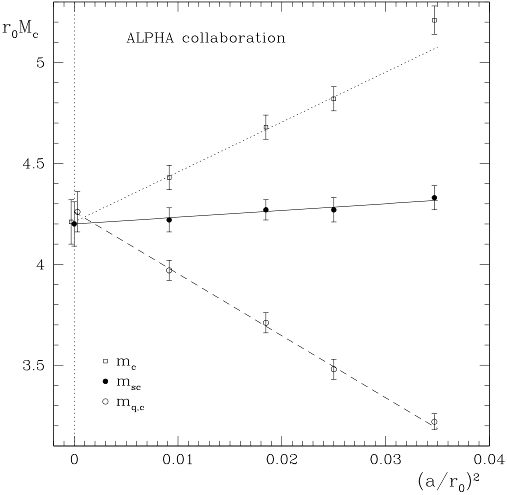
Example 2: pion mass in $N_\mathrm{f} = 2$ tmQCD

[ETM coll. Baron et al '09]



Example 3: O(a) improved charm quark mass (quenched)

[ALPHA coll. J. Rolf et al '02]



The 2d O(n) sigma model: a test laboratory for QCD

$$S = \frac{n}{2\gamma}\sum_{x,\mu}(\partial_\mu \mathbf{s})^2, \qquad \mathbf{s} = (s_1, \ldots, s_n) \qquad \mathbf{s}^2 = 1$$

- like QCD the model has a mass gap and is asymptotically free
- many analytical tools: large n expansion, Bethe ansatz, form factor bootstrap, etc.
- efficient numerical simulations due to cluster algorithms.

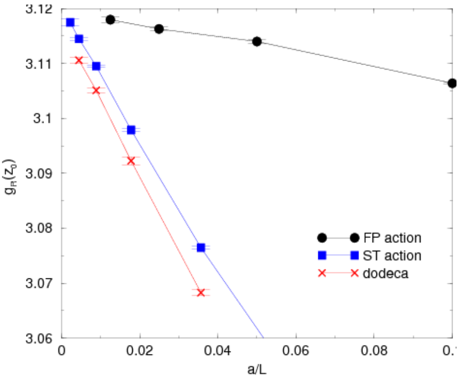
$\Rightarrow$ very precise data over a wide range of lattice spacing (a can be varied by 1-2 orders of magnitude).

- Symanzik: expect O(a^2) effects, up to logarithms
- Large n, at leading [Caracciolo, Pelissetto '98] and next-to-leading [Knechtli, Leder, Wolff '05]:

$$P(a) \sim P(0) + \frac{a^2}{L^2}(c_1 + c_2 \ln(a/L))$$

A sobering result (1):

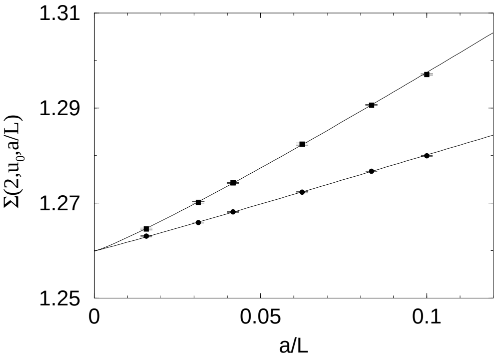
Numerical study of renormalised finite volume coupling to high precision ($n = 3$) [Hasenfratz, Niedermayer '00, Hasenbusch et al. '01, Balog et al. '09]



- Cutoff effects seem to be almost linear in a!
- Is this just an unfortunate case?

A sobering result (2):

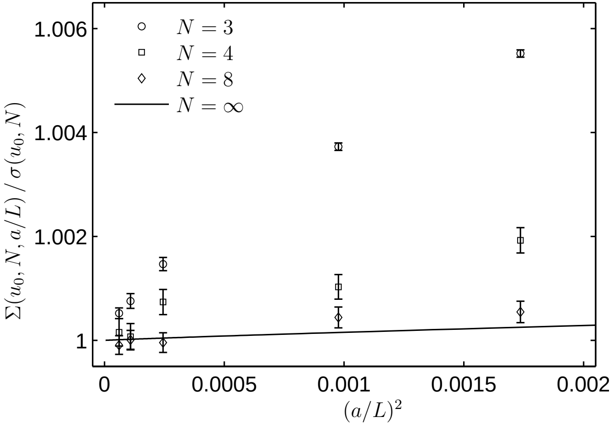
[Balog, Niedermayer & Weisz '09]



Fits with a and $a \ln a$ terms, lattice sizes $L/a = 10, \ldots, 64$

A closer look (1)

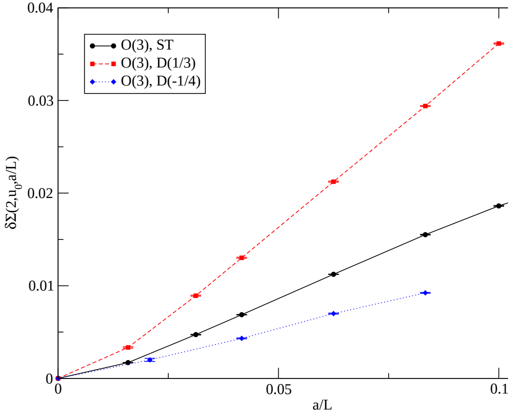
[Knechtli, Leder, Wolff '05], plot of cutoff effects vs. a^2/L^2, various n:



Asymptotic behaviour for larger n according to expectation, what about $n = 3$?

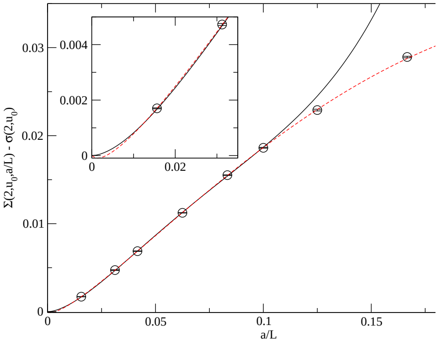
A closer look (2)

Continuum limit for mass gap $m(L)$ known analytically [Balog & Hegedus '04]! Subtract & study pure cutoff effect [Balog, Niedermayer, Weisz '09]



A closer look (3)

Continuum limit for mass gap $m(L)$ known analytically [Balog & Hegedus '04]!
Subtract & study pure cutoff effect: $\Sigma(2, u_0, a/L) - \sigma(2, u_0)$:



$c_1 a + c_2 a \ln a + c_3 a^2$ (dashed) vs. $c_1 a^2 + c_2 a^2 \ln a + c_3 a^4$ (solid)

A closer look (4) & solution of puzzle

[Balog, Niedermayer & Weisz '09]

- performed two-loop calculation with both effective Symanzik theory and lattice theory (various actions)
- Matching of both sides and subsequent RG considerations

$\Rightarrow$ Symanzik theory predicts for $\mathrm{O}(n)$ model leading $\mathrm{O}(a^2)$ behaviour:

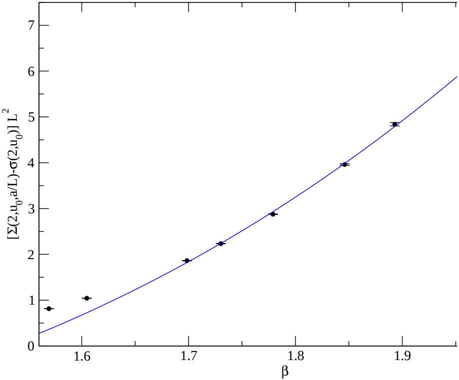
$$\delta(a) \propto a^2 \left(\ln a^2\right)^{n/(n-2)}$$

- compatible with large n result since $\lim_{n\to\infty}\{n/(n-2)\} = 1$
- For $\mathrm{O}(3)$ model:

$$\delta(a) \propto a^2 \left(\ln^3(a^2) + c_1 \ln^2(a^2) + c_2 \ln(a^2) + c_4\right) + \mathrm{O}(a^4)$$

A closer look (5)

Coefficient of $O(a^2)$ term [Balog, Niedermayer & Weisz '09]:



Not exactly constant! Multiplied with a^2 obtain "fake" linear behaviour in a!

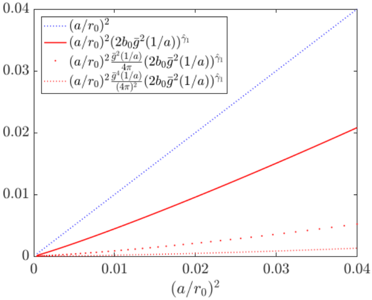
Yang-Mills theory and QCD

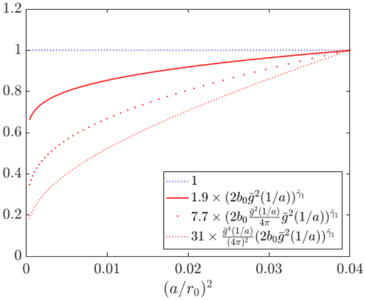
Recently, [N. Husung, P. Marquard and R. Sommer, 2020] have obtained the powers of the logs modifying the a^2-effects for spectral quantities (e.g. glueball masses) in pure SU(3) gauge theory (various lattice regularizations):

- find <u>negative</u> leading exponents of $(-7/11, -63/55)$ of $\ln(a)$!

$\Rightarrow$ Convergence to continuum is faster than a^2

- Work for QCD in progress, very interesting, keep tuned!





Summary, approach to the continuum

- Symanzik's analysis is applicable beyond perturbation theory
- Powers of a are accompanied by fractional powers of logarithms; these are perturbatively computable due to asymptotic freedom, using Symanzik's effective theory.
- Lesson from O(3) model: logarithmic corrections to powers in a can be large ($a^2 \ln^3(a)$);
  However, results by Husung et al. in pure gauge theory and QCD are re-assuring.
- It helps to combine results from different regularisations: renormalised quantities must have the same continuum limit
- Numerical results in QCD: typically lattice spacing varied by a factor 2, logarithms ignored (cannot be resolved).

$\Rightarrow$ Fitting higher order polynomials in a ignoring the logarithms has no theoretical basis and can be dangerous!