Non-perturbative Renormalization and Improvement of Lattice QCD

Lecture 2

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Contents

A remark on the scope of perturbation theory

The statement "QCD at low energies cannot be treated non-perturbatively due to the coupling being large" is only half of the truth:

- In QCD with massless quarks, all non-vanishing hadronic masses (protons, glueballs,...) are proportional to the Λ-parameter and thus <u>non-analytic</u> in the coupling!

$\Rightarrow$ not computable in perturbation theory whatever the size of the coupling!

- The chiral condensate is extracted from the 1-point function:

$$\Sigma = -\lim_{m_q \to 0} \lim_{a \to 0} \lim_{L \to \infty} \langle \left(\bar{\psi}(x)\psi(x)\right)_{\mathrm{R}} \rangle$$

- $\Sigma = 0$ exactly in perturbation theory, order by order!

Momentum Subtraction Schemes (MOM)

Recall procedure in continuum perturbation theory:

- example: renormalisation of the pseudoscalar density $P^a(x) = \overline{\psi}(x)\gamma_5 \frac{1}{2}\tau^a \psi(x)$:
- Correlation functions in momentum space with external quark states:

$$
\begin{aligned}
\left\langle \tilde{\psi}(p)\tilde{\overline{\psi}}(q) \right\rangle &= (2\pi)^4 \delta(p+q) S(p) \qquad \text{quark propagator} \\
\left\langle \tilde{\psi}(p)\tilde{P}^a(q)\tilde{\overline{\psi}}(p') \right\rangle &= (2\pi)^4 \delta(p+q+p') S(p) \Gamma_P^a(p,q) S(p+q)\,,
\end{aligned}
$$

- At tree-level:

$$
\begin{aligned}
\Gamma_P^a(p,q)|_{\mathrm{tree}} &= \gamma_5 \tfrac{1}{2}\tau^a, \\
\Rightarrow \quad \tfrac{1}{24}\sum_{a=1}^{3} \mathrm{tr}\,\left\{\gamma_5 \tau^a \Gamma_P^a(p,q)|_{\mathrm{tree}}\right\} &= 1
\end{aligned}
$$

- Renormalised fields:

$$\psi_{\mathrm{R}} = Z_{\psi}\psi, \qquad \overline{\psi}_{\mathrm{R}} = Z_{\psi}\overline{\psi}, \qquad P^a_{\mathrm{R}} = Z_{\mathrm{P}} P^a$$

$\Rightarrow$ renormalised vertex function:

$$\Gamma^a_{P,\mathrm{R}}(p,q) = Z_{\mathrm{P}} Z_{\psi}^{-2} \Gamma^a_P(p,q)$$

- typical MOM renormalisation condition (quark masses set to zero; mass-independent scheme!):

$$\Gamma^a_{P,\mathrm{R}}(p,0)\big|_{\mu^2=p^2} = \gamma_5 \tfrac{1}{2}\tau^a \qquad \Rightarrow \qquad Z_{\mathrm{P}} Z_{\psi}^{-2}$$

- equivalently using "projector":

$$\tfrac{1}{24}\sum_{a=1}^{3} \mathrm{tr}\,\left\{\gamma_5\tau^a\,\Gamma^a_{P,\mathrm{R}}(p,0)\big|_{\mu^2=p^2}\right\} = 1$$

- Determine Z_{ψ} either from quark propagator or use MOM scheme for vertex function of a conserved current

$$\Gamma_{V,\mathrm{R}}(p,q) = Z_{\psi}^{-2}\Gamma_V(p,q)$$

Summary: MOM schemes in the continuum

- Renormalisation condions are imposed on vertex functions in the gauge fixed theory with external quarks, gluons or ghosts.
- The vertex functions are taken in momentum space; a particular momentum configuration is chosen, such that the vertex function becomes a function of a single momentum p;
- MOM condition: a renormalised vertex function at subtraction scale $\mu^2 = p^2$ equals its tree-level expression,

$\Rightarrow$ mass independent scheme if quark masses $= 0$.

- Definition of renormalised gauge coupling: take vertex function of either the 3-gluon vertex, the quark-gluon vertex or the ghost-gluon vertex.
- Renormalisation constants depend on the chosen gauge! Require renormalisation of quark, gluon and ghost fields.

RI/MOM Schemes (RI = Regularisation Independent; MOM = Momentum Subtraction)

[Martinelli et al '95]: mimick the procedure in perturbation theory:

- choose Landau gauge, $\partial_\mu A_\mu = 0$; can be implemented on the lattice by a minimisation procedure
- RI/MOM schemes are very popular: many collaborations use it:
  - straightforward to implement on the lattice; many improvements over the years regarding algorithmic questions
  - can often be used on the very same gauge configurations which are produced for hadronic physics
- Regularisation Independence (RI) means: correlation functions of a renormalised operator do not depend on the regularisation used (up to cutoff effects).

RI/MOM schemes, discussion

- Suppose we have calculated a renormalised hadronic matrix element of the multiplicatively renormalisable operator O

$$\mathcal{M}_O(\mu) = \lim_{a \to 0} \langle h | O_{\mathrm{R}}(\mu) | h' \rangle$$

- Provided μ is in the perturbative regime, one may evaluate the MOM scheme in continuum perturbation theory and evolve to a different scale:

$$\begin{aligned} \mathcal{M}_O(\mu') &= U(\mu', \mu) \mathcal{M}_O(\mu), \\ U(\mu', \mu) &= \exp \left\{ \int_{\bar{g}(\mu)}^{\bar{g}(\mu')} \frac{\gamma_O(g)}{\beta(g)} \mathrm{d}g \right\} \end{aligned}$$

- N.B. Continuum perturbation theory is available to 3-loops in some cases!

RI/MOM schemes, what could go wrong?

- The scale μ could be too low; need to hope for a "window"

$$\Lambda_{\mathrm{QCD}} \ll \mu \ll a^{-1}$$

In practice scales are often too low: non-perturbative effects (e.g. pion poles, condensates) are then eliminated by fitting to expected functional form (from OPE in fixed gauge);

$\Rightarrow$ errors are difficult to quantify!

- Gribov copies: the (Landau) gauge condition does not have a unique solution on the full gauge orbit
- Perturbative calculations are made using
  - infinite volume
  - vanishing quark masses

$\Rightarrow$ difficult for numerical simulations especially in full QCD.

A prominent non-perturbative effect: the pion pole

[Martinelli et al. '95]

- Consider the 3-point correlation function for P^a used in MOM condition:

$$\int \mathrm{d}^4x \int \mathrm{d}^4y\, \mathrm{e}^{-ipx} \langle \overline{\psi}(0)\gamma_5 \tfrac{1}{2}\tau^b \psi(x) P^a(y)\rangle$$

- The contribution for $x \approx 0$ is proportional to the pion propagator at vanishing momentum:

$$\int \mathrm{d}^4y \langle P^b(0) P^a(y)\rangle \propto \left.\frac{1}{m_\pi^2 + q^2}\right|_{q^2=0}$$

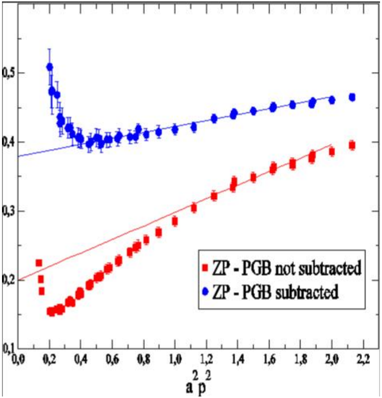
- Insertion into the MOM condition at $\mu^2 = p^2$ yields

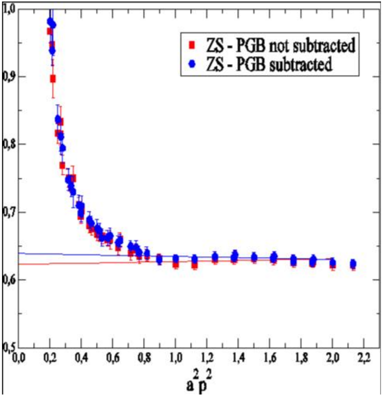
$$Z_{\mathrm{P}}^{\mathrm{MOM}} \sim \frac{A}{\mu^2 m_\pi^2} + \ldots$$

$\Rightarrow$ the chiral limit is ill-defined, as $m_\pi^2 \propto m_q$ (cf. lectures on χPT)!

RI/MOM scheme, example 1

[ETMC collaboration, talk by P. Dimopoulos at Lattice '07] twisted mass QCD with $N_\mathrm{f} = 2$, subtraction of pion pole à la [Giusti, Vladikas '00]





While Z_S shows the expected plateau, Z_P shows some slope even after subtraction of the pion pole (cutoff effects?)

A prominent non-perturbative effect: the pion pole

- pion poles; in the original MOM scheme the problem is most severe due to the kinematical choice $q = 0$

$\Rightarrow$ $q = 0$ originally dictated by available continuum perturbation theory!. Requires a subtraction of this effect!

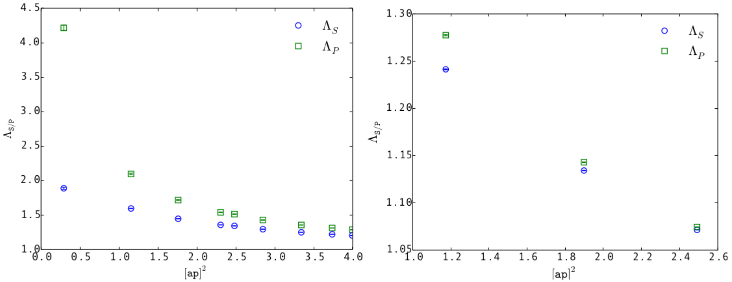
- Major improvement: RI/SMOM scheme [Aoki et al. '08; Sturm et al. '09]): quark momenta p and p' and the vertex momentum q are taken equal in magnitude but non-zero (non-exceptional momentum configuration) The pion pole contribution becomes

$$Z_{\mathrm{P}}^{\mathrm{SMOM}} \sim \frac{A}{\mu^2(\mu^2 + m_\pi^2)}$$

and is finite in the chiral limit $m_\pi = 0$ and suppressed by μ^4 rather than μ^2.

RI/SMOM scheme, example

[A. Lytle (HPQCD) '15] quark mass renormalization with HISQ:

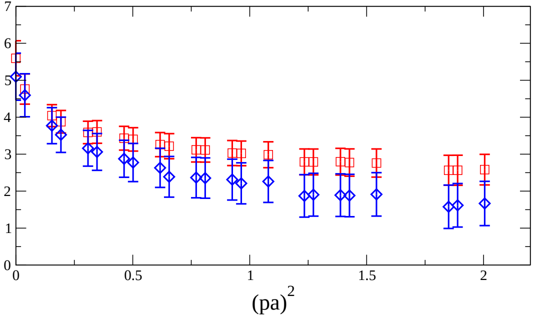


Comparison between RI/MOM (left) and RI/SMOM (right) for scalar and pseudo-scalar vertex functions

$\Rightarrow$ Much reduced sensitivity to pion pole!

RI/MOM scheme, example 2

[Huey-Wen Lin '06] study of quark gluon vertex:



- Comparison of the quark vertex function in Landau gauge, fixed in two different ways on the same ensemble of gauge configurations
- Influence of Gribov copies can be sizable!

RI/MOM schemes; Summary

- RI/(S)MOM schemes are widely used and have allowed for many successful applications!
- Non-perturbative effects like the pion pole are either subtracted or taken into account by fits to the expected p^2-behaviour; error estimates are difficult!
- A warning from the quark-gluon vertex: the effect of Gribov copies should be monitored!
- Finite volume and quark mass effects are often found to be small.
- Since the method can be applied at little cost on the existing configurations for hadronic physics, it can always be tried!

Improvement of RI/MOM schemes

Many technical improvements over the last 20 years:

- RI/SMOM scheme (non-exceptional momenta): reduces the problem with Goldstone poles; Many continuum perturbation theory results now available up to 3-loop order.
- Can one reach higher scales? Small steps in scale may be feasible [Arthur & Boyle '10]; in principle need to promote to finite volume scheme: set $\mu = 1/L$, however:
  - periodic b.c's: need gauge fixing on the torus (complicated)
  - twisted gauge field b.c.'s? constraints on N_c and N_f
  - perturbation theory to be re-done from scratch (finite volume is part of the scheme!)
- use gauge invariant correlation functions $\Rightarrow$ no Gribov copies; PT more difficult, larger cutoff effects?

Finite volume schemes, requirements

Wanted: renormalization scheme which

- is defined in a finite space-time volume
- is non-perturbatively defined;
- can be expanded in perturbation theory (up to 2-loop) with reasonable effort;
- is gauge invariant;
- is quark mass-independent.
- can be evaluated by numerical simulation!

$\Rightarrow$ use the Schrödinger functional!

The Schrödinger functional (formal continuum)

The Schrödinger functional appears naturally in the Schrödinger representation of QFT (Symanzik '81), as the time evolution kernel when integrating the functional Schrödinger equation:
Wave functional in Dirac's notation (A, A': field configurations at (Euclidean) times 0, T):

$$\begin{aligned} \psi[A] &\equiv \langle A|\psi\rangle \\ \psi'[A'] &= \int D[A]\langle A'|\mathrm{e}^{-T\mathbb{H}}|A\rangle\langle A|\psi\rangle \end{aligned}$$

The Schrödinger functional is a functional of the initial and final field configuration:

$$\mathcal{Z}[A, A'] = \langle A'|\mathrm{e}^{-T\mathbb{H}}|A\rangle = \int D[\phi]\mathrm{e}^{-S}.$$

The Euclidean field ϕ satisfies Dirichlet boundary conditions

$$\phi(x)|_{x_0=0} = A(\mathbf{x}) \qquad \phi(x)|_{x_0=T} = A'(\mathbf{x})$$

The Schrödinger functional is an example of a field theory defined on a manifold with boundary $\Rightarrow$ problems/questions:

- Translation invariance is broken $\Rightarrow$ momentum is not conserved.
- Conventional proofs of perturbative renormalisability rely on power counting theorems in momentum space: not applicable here!
- Heuristic arguments by Symanzik:

A renormalisable QFT remains renormalisable when considered on a manifold with boundary. Besides the usual parameter and field renormalisations one just needs to add a complete set of local boundary counterterms to the action, i.e. polynomials in the fields and its derivatives of dimension 3 or less, integrated over the boundary.
Scalar ϕ_4^4-theory, boundary at $x_0 = 0$:

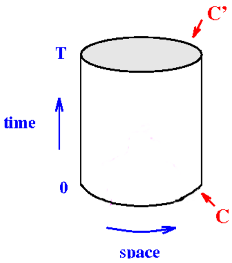
$$\int_{x_0=0} \mathrm{d}^3\mathbf{x}\,\phi^2, \qquad \int_{x_0=0} \mathrm{d}^3\mathbf{x}\,\phi\partial_0\phi$$

The Schrödinger functional in QCD (formal continuum)

Definition for gauge theories and QCD is analogous: The Schrödinger functional is the functional integral on a hyper cylinder,

$$\mathcal{Z} = \int_{\text{fields}} \mathrm{e}^{-S}$$

with periodic boundary conditions in spatial directions and Dirichlet conditions in time.



Boundary conditions for gluon and quark fields:

$P_\pm = \frac{1}{2}(1 \pm \gamma_0)$,

$$\begin{aligned} P_+\psi(x)|_{x_0=0} &= \rho & \qquad P_-\psi(x)|_{x_0=T} &= \rho' \\ \overline{\psi}(x)P_-|_{x_0=0} &= \bar{\rho} & \qquad \overline{\psi}(x)P_+|_{x_0=T} &= \bar{\rho}', \\ A_k(x)|_{x_0=0} &= C_k & \qquad A_k(x)|_{x_0=T} &= C'_k \end{aligned}$$

Correlation functions are then defined as usual

$$\langle O \rangle = \left\{ Z^{-1} \int_{\text{fields}} O \, \mathrm{e}^{-S} \right\}_{\rho=\rho'=0;\, \bar{\rho}=\bar{\rho}'=0}$$

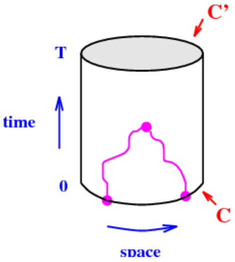
O may contain quark boundary fields

$$\zeta(\mathbf{x}) \equiv P_{-}\zeta(\mathbf{x}) = \frac{\delta}{\delta\bar{\rho}(\mathbf{x})}$$

$$\bar{\zeta}(\mathbf{x}) \equiv \bar{\zeta}(\mathbf{x})P_{+} = -\frac{\delta}{\delta\rho(\mathbf{x})}$$

$$\zeta'(\mathbf{x}) \equiv P_{+}\zeta'(\mathbf{x}) = \frac{\delta}{\delta\bar{\rho}'(\mathbf{x})}$$

$$\bar{\zeta}'(\mathbf{x}) \equiv \bar{\zeta}'(\mathbf{x})P_{+} = -\frac{\delta}{\delta\rho'(\mathbf{x})}$$



$\Rightarrow$ the boundary values of the quark fields are used as external sources

Properties of the QCD Schrödinger functional

- The SF is renormalisable: as usual for coupling and quark masses; the boundary quark fields require a multiplicative renormalisation.
- absence of fermionic zero modes: numerical simulations at zero quark masses are possible!
- For some choices of C_k and C'_k it can be shown that the induced background gauge field is an absolute minimum of the action $\Rightarrow$ perturbation theory is straightforward and seems practical at least to 2-loop order.
- As C_k and C'_k are held fixed only spatially constant gauge transformations are possible at the boundaries!:

$$C_k(\mathbf{x}) \to \Lambda(\mathbf{x}) C_k(\mathbf{x}) \Lambda^{-1}(\mathbf{x}) + \Lambda(\mathbf{x}) \partial_k \Lambda^{-1}(\mathbf{x})$$

i.e. the allowed $\Lambda(\mathbf{x}) \in \mathrm{SU}(N)$ must be $\mathbf{x}$-independent and commute with C_k.

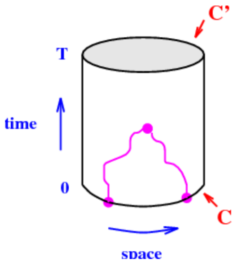
- Therefore, bilinear boundary quark sources such as

$$\mathcal{O}^a = \int \mathrm{d}^3\mathbf{y}\mathrm{d}^3\mathbf{z}\; \bar{\zeta}(\mathbf{y})\gamma_5 \tfrac{\tau^a}{2}\zeta(\mathbf{z}), \qquad \mathcal{O}'^a = \int \mathrm{d}^3\mathbf{y}\mathrm{d}^3\mathbf{z}\; \bar{\zeta}'(\mathbf{y})\gamma_5 \tfrac{\tau^a}{2}\zeta'(\mathbf{z})$$

  are gauge invariant!

- Typical gauge invariant correlation functions are then

$$f_{\mathrm{P}}(x_0) = -\tfrac{1}{3}\sum_{a=1}^{3}\langle P^a(x)\mathcal{O}^a\rangle, \qquad f_{\mathrm{A}}(x_0) = -\tfrac{1}{3}\sum_{a=1}^{3}\langle A_0^a(x)\mathcal{O}^a\rangle,$$



$\Rightarrow$ convenient in perturbation theory: in contrast to a periodic or infinite volume where gauge invariant fermionic correlation functions lead to one-loop diagrams at lowest order, e.g.

$$g_{\mathrm{PP}}(x_0) = -a^3 \sum_{\mathbf{x}} \sum_{a=1}^{3} \langle P^a(x) P^a(0) \rangle$$

- dimensional analysis $\Rightarrow$ at short distances one finds the asymptotic behaviour (up to logarithms):

$$g_{\mathrm{PP}}(x_0) \sim \frac{\mathrm{const}}{(x_0)^3}, \qquad f_{\mathrm{P}}(x_0) \sim \mathrm{const}$$

- expect
  - small cutoff effects for $f_{\mathrm{P}}(x_0)$ due to mild x_0-dependence
  - good signal in numerical simulations.

More on the renormalisability of the SF

- no gauge invariant dimension ≤ 3 counterterm exists, the pure gauge SF is finite after renormalisation of the coupling constant
- continuum quark action with SF boundary conditions at tree-level:

$$S_{\mathrm{f}} = \int \mathrm{d}^4 x \, \overline{\psi} \left(\tfrac{1}{2} \overleftrightarrow{\not{D}} + m \right) \psi - \tfrac{1}{2} \int_{x_0=0} \mathrm{d}^3 \mathbf{x} \, \overline{\psi} \psi - \tfrac{1}{2} \int_{x_0=T} \mathrm{d}^3 \mathbf{x} \, \overline{\psi} \psi$$

Exercise:

Show that the boundary terms are necessary if one requires the existence of smooth solutions to the equations of motion with SF boundary conditions

- The counterterms are linear in the boundary fields

$$\begin{aligned} \overline{\psi}(x)\psi(x)|_{x_0=0} &= \bar{\rho}(\mathbf{x}) P_- \psi(0,\mathbf{x}) + \overline{\psi}(0,\mathbf{x}) P_+ \rho(\mathbf{x}), \\ \overline{\psi}(x)\psi(x)|_{x_0=T} &= \bar{\rho}'(\mathbf{x}) P_+ \psi(T,\mathbf{x}) + \overline{\psi}(T,\mathbf{x}) P_- \rho'(\mathbf{x}), \end{aligned}$$

More on the renormalisability of the SF

- The only dimension 3 counterterm with correct symmetries is $\overline{\psi}\psi$
- Time reversal symmetry requires the same coefficient at $x_0 = 0, T$
- This counterterm can thus be absorbed in a multiplicative rescaling of $\rho, \rho', \bar{\rho}, \bar{\rho}'$ by the same renormalization constant:

$$\rho_{\mathrm{R}} = Z_\rho \rho, \qquad \bar{\rho}_{\mathrm{R}} = Z_\rho \bar{\rho}, \qquad \rho'_{\mathrm{R}} = Z_\rho \rho', \qquad \bar{\rho}'_{\mathrm{R}} = Z_\rho \bar{\rho}'$$

  Consequently, setting $Z_\zeta = Z_\rho^{-1}$:

$$\zeta_{\mathrm{R}} = Z_\zeta \zeta, \qquad \zeta'_{\mathrm{R}} = Z_\zeta \zeta', \qquad \overline{\zeta}_{\mathrm{R}} = Z_\zeta \overline{\zeta}, \qquad \overline{\zeta}'_{\mathrm{R}} = Z_\zeta \overline{\zeta}',$$

- Hence sources like $\mathcal{O}^a$ are multiplicatively renormalised by Z_ζ^2

Definition of the SF coupling [Lüscher et al. '92]

- Choose abelian and spatially constant boundary gauge fields:

$$C_k = \frac{i}{L}\begin{pmatrix}\phi_1 & 0 & 0\\ 0 & \phi_2 & 0\\ 0 & 0 & \phi_3\end{pmatrix}, \qquad C'_k = \frac{i}{L}\begin{pmatrix}\phi'_1 & 0 & 0\\ 0 & \phi'_2 & 0\\ 0 & 0 & \phi'_3\end{pmatrix}, \qquad k = 1,2$$

- with angles taken to be linear functions of a parameter η:

$$\begin{aligned} \phi_1 &= \eta - \tfrac{\pi}{3}, & \phi'_1 &= -\phi_1 - \tfrac{4\pi}{3},\\ \phi_2 &= -\tfrac{1}{2}\eta, & \phi'_2 &= -\phi_3 + \tfrac{2\pi}{3},\\ \phi_3 &= -\tfrac{1}{2}\eta + \tfrac{\pi}{3}, & \phi'_3 &= -\phi_2 + \tfrac{2\pi}{3}. \end{aligned}$$

- The gauge action has an absolute minimum for:

$$B_0 = 0, \qquad B_k = \left[x_0 C'_k + (L - x_0) C_k\right]/L, \quad k = 1,2,3.$$

i.e. other gauge fields with the same action must be gauge equivalent to B_μ

Definition of the SF coupling

- Define the effective action of the induced background field

$$\Gamma[B] = -\ln \mathcal{Z}[C, C']$$

- In perturbation theory the effective action has the expansion

$$\Gamma[B] \quad \sim \quad g_0^{-2}\Gamma_0[B] + \Gamma_1[B] + O(g_0^2)$$

- Definition of the SF coupling:

$$\bar{g}^2(L) = \left.\frac{\partial_\eta \Gamma_0[B]|_{\eta=0}}{\partial_\eta \Gamma[B]|_{\eta=0}}\right|_{m_{\mathrm{q,i}}=0} \qquad \Rightarrow \quad \bar{g}^2(L) = g_0^2 + \mathrm{O}(g_0^4)$$

- b.c.'s induce a constant colour electric field:

$$G_{0k} = \partial_0 B_k = \frac{C_k - C'_k}{L}$$

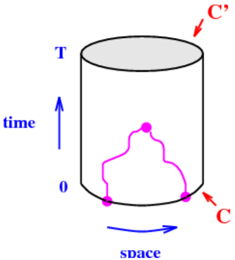
$\Rightarrow$ The coupling is defined as "response coefficient" to a variation of a constant colour electric field.

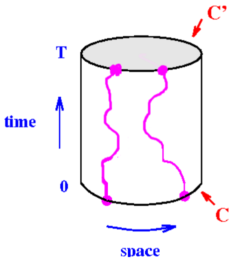
Renormalisation of operators in the SF scheme (1)

Example: renormalisation of $P^a = \overline{\psi}\gamma_5 \frac{\tau^a}{2}\psi$:

- In this case we set $C_k = C'_k = 0$, i.e. trivial background field $B = 0$
- Define correlation functions

$$f_{\rm P}(x_0) = -\tfrac{1}{3}\langle \mathcal{O}^a P^a(x)\rangle, \qquad f_1 = -\tfrac{1}{3L^6}\langle \mathcal{O}^a \mathcal{O}'^a\rangle$$





Renormalisation of operators in the SF scheme (2)

- Renormalised correlation functions:

$$f_{\mathrm{P,R}}(x_0) = Z_\zeta^2 Z_P f_{\mathrm{P}}(x_0), \qquad f_{1,R} = Z_\zeta^4 f_1,$$

  set $T = L$, $m = 0$, $x_0 = L/2$, and impose

$$Z_{\mathrm{P}}(g_0, L/a) \frac{f_{\mathrm{P}}(L/2)}{\sqrt{f_1}} = \left. \frac{f_{\mathrm{P}}(L/2)}{\sqrt{f_1}} \right|_{g_0=0}$$

- similarity with MOM schemes: the renormalised amplitude at $\mu = L^{-1}$ equals its tree-level expression
- The ratio is formed to cancel any Z_ζ.
- definition of running quark mass: $\overline{m}(L) = Z_{\mathrm{P}}^{-1}(L)m$.

Step Scaling Functions

- The aim is to construct the Step Scaling Functions $\sigma(u)$ and $\sigma_{\mathrm{P}}(u)$:

$$\begin{aligned}\sigma(u) &= \bar{g}^2(2L)\big|_{u=\bar{g}^2(L)}, \\ \sigma_{\mathrm{P}}(u) &= \lim_{a\to 0} \frac{Z_{\mathrm{P}}(g_0, 2L/a)}{Z_{\mathrm{P}}(g_0, L/a)}\bigg|_{u=\bar{g}^2(L)}\end{aligned}$$

- These are related to the usual RG functions:

$$\int_{\sqrt{\sigma(u)}}^{\sqrt{u}} \frac{\mathrm{d}g}{\beta(g)} = \ln 2 \qquad \sigma_{\mathrm{P}}(u) = \exp \int_{\sqrt{\sigma(u)}}^{\sqrt{u}} \frac{\tau(g)}{\beta(g)} \mathrm{d}g$$

- One thus considers a change of scale by a finite factor $s = 2$; RG functions tell us what happens for infinitesimal scale changes.

Lattice approximants $\Sigma(u, a/L)$ for $\sigma(u)$

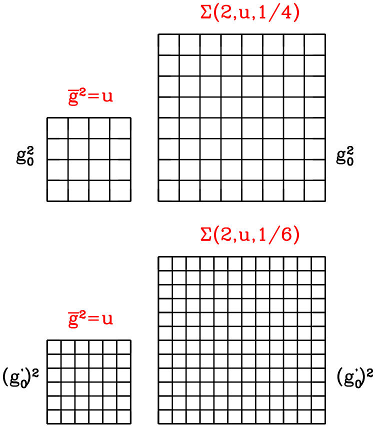
- choose g_0 and $L/a = 4$, measure $\bar{g}^2(L) = u$ (this sets the value of u)
- double the lattice and measure

$$\Sigma(u, 1/4) = \bar{g}^2(2L)$$

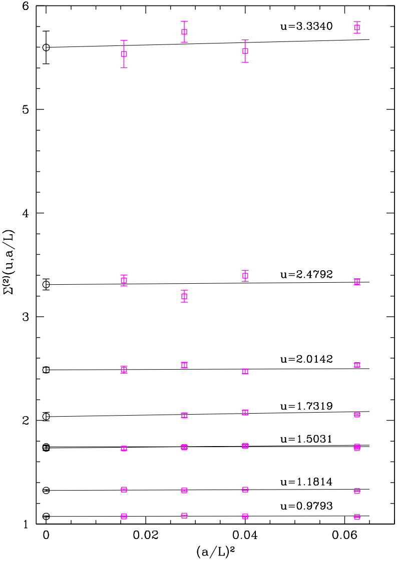
- now choose $L/a = 6$ and tune g_0' such that $\bar{g}^2(L) = u$ is satisfied
- double the lattice and measure

$$\Sigma(u, 1/6) = \bar{g}^2(2L)$$

- and so on ...

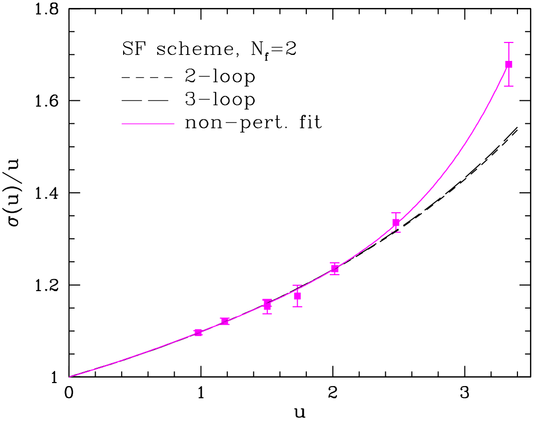


Continuum extrapolation of the SSF [ALPHA '05]



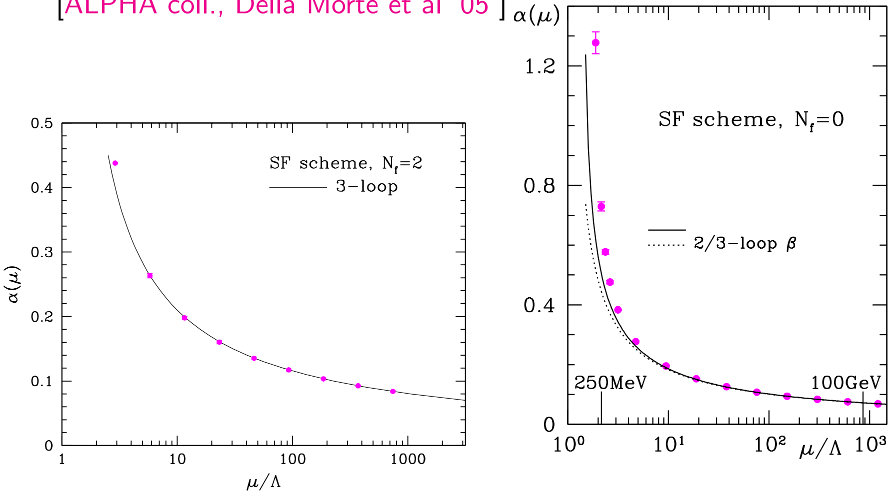
The SSF in the continuum limit

[ALPHA coll., Della Morte et al '05]



The running of the SF coupling

[ALPHA coll., Della Morte et al '05]



Determination of the Λ-parameter

- The formula

$$\Lambda = \mu \left(b_0 \bar{g}^2\right)^{-b_1/2b_0^2} \exp\left\{-\frac{1}{2b_0\bar{g}^2}\right\} \times \exp\left\{-\int_0^{\bar{g}} \mathrm{d}x \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\}$$

  holds for any value of μ. We may use it at L_{min} to obtain

$$\Lambda L_{\mathrm{min}} = f\left(\bar{g}(L_{\mathrm{min}})\right)$$

- The function $f(g)$ can be evaluated at $g = \bar{g}(L_{\mathrm{min}})$ since this is deep in the perturbative region. The integral in the exponent

$$\int_0^{\bar{g}} \mathrm{d}x \left[\frac{b_2 b_0 - b_1^2}{b_0^3} x + \mathrm{O}(x^3)\right] = \frac{b_2 b_0 - b_1^2}{2b_0^3} \bar{g}^2 + \mathrm{O}(\bar{g}^4)$$

  may thus be evaluated using the β-function at 3-loop order.
- Since $L_{\mathrm{max}} = 2^n L_{\mathrm{min}}$ one knows $L_{\mathrm{max}} \Lambda$
- still need $F_\pi L_{\mathrm{max}}$

Matching to a low energy scale

Ideally one would like to compute e.g. $F_\pi \Lambda$, and take $F_\pi = 132\mathrm{MeV}$ from experiment

- What is required? The scale L_{max} is implicitly defined:

$$\bar{g}^2(L_{\mathrm{max}}) = 4.84 \qquad \Rightarrow \qquad (L_{\mathrm{max}}/a)(g_0)$$

  Setting $L_{\mathrm{max}}/a = 6, 8, 10, \ldots$ one then finds corresponding values of the bare coupling (at fixed g_0 some interpolation of L_{max}/a will be necessary instead)

- One must then be able to compute aF_π in a large volume simulation at the very same values of the bare coupling:

$$F_\pi \Lambda = \lim_{g_0 \to 0} (L_{\mathrm{max}}/a)(g_0)(aF_\pi)(g_0)$$

- One thus needs a range of g_0 where both can be computed, aF_π and $\bar{g}(L_{\mathrm{max}})$
- Remark: intermediate results are often quoted in terms of Sommer's scale r_0, rather than F_π.

Results for QCD with $N_\mathrm{f} = 0$ and $N_\mathrm{f} = 2$ quark flavours

- The scale r_0 [R. Sommer '93] is obtained from the force $F(r)$ between static quark and antiquark separated by a distance r:

$$r_0^2 F(r_0) = 1.65$$

  The r.h.s. was chosen so that phenomenological estimates from potential models yield $r_0 = 0.5\,\mathrm{fm}$.

- Recent result for $N_\mathrm{f} = 2$ ([ALPHA '12]): $F_K = 155\,\mathrm{MeV}$ implies $r_0 = 0.503(10)\,\mathrm{fm}$ (at physical pion mass!).
- Results for Λ using $r_0 = 0.5\,\mathrm{fm}$ [ALPHA '99-'12]

$$\begin{aligned} \Lambda^{(2)}_{\overline{\mathrm{MS}}} r_0 &= 0.789(52), \qquad \Lambda^{(2)}_{\overline{\mathrm{MS}}} = 310(20)\,\mathrm{MeV} \\ \Lambda^{(0)}_{\overline{\mathrm{MS}}} r_0 &= 0.602(48), \qquad \Lambda^{(0)}_{\overline{\mathrm{MS}}} = 238(19)\,\mathrm{MeV} \end{aligned}$$

The running quark mass

- Coupled evolution of the running mass and the coupling:

$$
\begin{aligned}
\overline{m}(2L) &= \sigma_m(u)\overline{m}(L), \qquad \sigma_m(u) = 1/\sigma_{\mathrm{P}} \\
\bar{g}^2(2L) &= \sigma(u)
\end{aligned}
$$

- Once the running coupling is known in a range $[u_0, u_n]$,

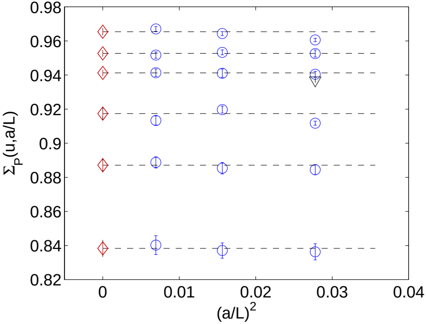
$$u_0 = \bar{g}^2(L_{\mathrm{min}}), \quad u_k = \bar{g}^2(2^k L_{\mathrm{min}}), k = 1, 2, \ldots, n$$

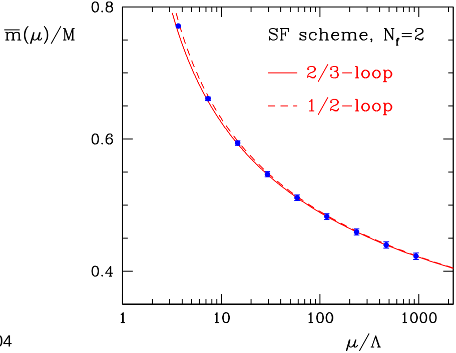
  determine $\sigma_m(u)$ for the same range of couplings: evolution of quark mass and coupling recursively

$$\overline{m}(2^k L_{\mathrm{min}})/\overline{m}(2^{k-1} L_{\mathrm{min}}) = \sigma_m(u_k), \qquad k = 1, 2, \ldots, n$$

- one obtains $\overline{m}(2L_{\mathrm{max}})/\overline{m}(L_{\mathrm{min}})$
- Extract $\overline{m}(L_{\mathrm{min}})/M$ using PT as for Λ-parameter

Running mass in the SF scheme [ALPHA '05]





Relation to bare quark masses

- In practice with Wilson type quarks, one avoids the additive renormalisation of the bare quark mass parameter by replacing it by a *measured* bare mass m_{PCAC} from the (bare) PCAC relation:

$$m_{\mathrm{PCAC}} \stackrel{\mathrm{def}}{=} \frac{\langle \partial_\mu A_\mu^a(x) O \rangle}{2\langle P^a(x) O \rangle}$$

- The running quark mass is then related to m_{PCAC}

$$\overline{m}(L) = \underbrace{Z_{\mathrm{P}}^{-1}(g_0, L/a) Z_{\mathrm{A}}(g_0)}_{\text{known factors}} \underbrace{m_{\mathrm{PCAC}}(g_0)}_{\text{measured}},$$

- Combine results,

$$M = Z_M(g_0) m_{\mathrm{PCAC}}(g_0)$$

and take the continuum limit $g_0 \to 0$.

Strange quark mass

The most recent $N_{\rm f} = 2$ result for the strange quark mass using this strategy ([ALPHA '12]):

$$M_s = 138(3)(1)\,\mathrm{MeV} \quad \Rightarrow \quad \overline{m}^{\overline{\mathrm{MS}}}(\mu = 2\,\mathrm{GeV}) = 102(3)(1)\,\mathrm{MeV}$$

- Quoted errors are statistical and systematic;
- Note: Except for quenching of the strange quark the ALL systematic errors have been addressed!

Concluding remarks

- The recursive finite volume technique has completely eliminated the problem with large scale differences.
- Physical results require a matching calculation at a low energy scale: it is crucial to have a range in bare couplings where both, the renormalisation conditions and the hadronic input can be computed
- Whether perturbation theory for the running coupling/operator is working well or not down to low scales is not so important; you would not know this beforehand!
- Most recent state of the art: QCD with $N_\mathrm{f} = 3$ & some further technical refinements:
    - gradient flow coupling in finite volume scheme at low energies
    - Global fits to the step-scaling function
    - Improved control of cutoff effects
    - use of the flow time scale t_0 instead of r_0
- Many more applications in the literature: all quark bilinear operators, $\Delta F = 2$ 4-quark operators, HQET,..