Perturbative tests at high energies, using lattice results by the ALPHA collaboration

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ALPHA Collaboration

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References, α_s by the ALPHA collaboration:

- "**Determination of the QCD Λ-parameter and the accuracy of perturbation theory at high energies**,"
  Mattia Dalla Brida, Patrick Fritzsch, Tomasz Korzec, Alberto Ramos, S.S., Rainer Sommer [ALPHA Collaboration],
  Phys. Rev. Lett. **117**, no. 18, 182001 (2016) arXiv:1604.06193 [hep-ph].
- "**A non-perturbative exploration of the high energy regime in $N_{\rm f} = 3$ QCD** ,"
  Mattia Dalla Brida, Patrick Fritzsch, Tomasz Korzec, Alberto Ramos, S.S., Rainer Sommer [ALPHA Collaboration],
  Eur. Phys. J. C **78** (2018) no.5, 372 arXiv:1803.10230 [hep-lat].
- "**Slow running of the Gradient Flow coupling from 200 MeV to 4 GeV in $N_{\rm f} = 3$ QCD,**"
  Mattia Dalla Brida, Patrick Fritzsch, Tomasz Korzec, Alberto Ramos, S.S, Rainer Sommer [ALPHA Collaboration],
  Phys. Rev. D **95**, no. 1, 014507 (2017), arXiv:1607.06423 [hep-lat].

$\Rightarrow$ "**QCD Coupling from a Nonperturbative Determination of the Three-Flavor Λ Parameter**,"
Mattia Bruno, Mattia Dalla Brida, Patrick Fritzsch, Tomasz Korzec, Alberto Ramos, Stefan Schaefer, S. S., Hubert Simma Rainer Sommer [ALPHA Collaboration],
Phys. Rev. Lett. **119**, no. 10, 102001 (2017), arXiv:1706.03821 [hep-lat].

Topics:

- Results for the SF coupling between $1/L_0 \approx 4\mathrm{GeV}$ and O(100) GeV
- Extraction of $L_0\Lambda^{(3)}$ & tests of perturbation theory
- Summary

The QCD Λ-parameter vs. $\alpha_s(\mu) = \bar{g}^2(\mu)/4\pi$

The coupling $\alpha_s(\mu)$ can be traded for its associated Λ-parameter:

$$\Lambda = \mu\varphi(\bar{g}(\mu)) = \mu\left[b_0\bar{g}^2(\mu)\right]^{-\frac{b_1}{2b_0^2}} \mathrm{e}^{-\frac{1}{2b_0\bar{g}^2(\mu)}} \exp\left\{-\int_0^{\bar{g}(\mu)} dg\left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\}$$

- exact solution of Callan-Symanzik equation: $\left(\mu\frac{\partial}{\partial\mu} + \beta(\bar{g})\frac{\partial}{\partial\bar{g}}\right)\Lambda = 0$
- Number N_{f} of massless quarks is fixed.
- If the coupling $\bar{g}(\mu)$ non-perturbatively defined so is its β-function!
- $\beta(g)$ has asymptotic expansion $\beta(g) = -b_0 g^3 - b_1 g^5 - b_2 g^7 ..$

  $$b_0 = (11 - \tfrac{2}{3}N_{\mathrm{f}})/(4\pi)^2, \qquad b_1 = (102 - \tfrac{38}{3}N_{\mathrm{f}})/(4\pi)^4, \quad \ldots$$

  $b_{0,1}$ are universal, scheme-dependence starts with 3-loop coefficient b_2.
- Scheme dependence of Λ almost trivial:

  $$g_{\mathrm{X}}^2(\mu) = g_{\mathrm{Y}}^2(\mu) + c_{\mathrm{XY}} g_{\mathrm{Y}}^4(\mu) + \ldots \quad \Rightarrow \quad \frac{\Lambda_{\mathrm{X}}}{\Lambda_{\mathrm{Y}}} = \mathrm{e}^{c_{\mathrm{XY}}/2b_0}$$

$\Rightarrow$ can use $\Lambda_{\overline{\mathrm{MS}}}$ as reference (even though the $\overline{\mathrm{MS}}$-scheme is purely perturbative!)

A family of SF couplings I

Dirichlet b.c.'s in Euclidean time, abelian boundary values C_k, C'_k:

$$A_k(x)|_{x_0=0} = C_k(\eta,\nu), \qquad A_k(x)|_{x_0=L} = C'_k(\eta,\nu)$$

$$C_k = \frac{i}{L}\begin{pmatrix} \eta - \frac{\pi}{3} & 0 & 0 \\ 0 & \eta\nu - \frac{\eta}{2} & 0 \\ 0 & 0 & -\eta\nu - \frac{\eta}{2} + \frac{\pi}{3} \end{pmatrix}, \quad C'_k = \frac{i}{L}\begin{pmatrix} -\eta - \pi & 0 & 0 \\ 0 & \eta\nu + \frac{\eta}{2} + \frac{\pi}{3} & 0 \\ 0 & 0 & \frac{\eta}{2} - \eta\nu + \frac{2\pi}{3} \end{pmatrix}$$

$\Rightarrow$ induce family of abelian, spatially constant background fields B_μ with parameters η, ν ($\rightarrow$ 2 abelian generators of SU(3)):

$$B_k(x) = C_k(\eta,\nu) + \frac{x_0}{L}\left(C'_k(\eta,\nu) - C_k(\eta,\nu)\right), \qquad B_0 = 0.$$

- Induced background field is unique up to gauge equivalence
- Effective action

$$\mathrm{e}^{-\Gamma[B]} = \int D[A,\psi,\overline{\psi}]\mathrm{e}^{-S[A,\psi,\overline{\psi}]}, \qquad \Gamma[B] = \frac{1}{g_0^2}\Gamma_0[B] + \Gamma_1[B] + \mathrm{O}(g_0^2)$$

- Define

$$\frac{1}{\bar{g}_\nu^2(L)} = \left.\frac{\partial_\eta \Gamma[B]}{\partial_\eta \Gamma_0[B]}\right|_{\eta=0} = \left.\frac{\langle \partial_\eta S\rangle}{\partial_\eta \Gamma_0[B]}\right|_{\eta=0}$$

$\Rightarrow$ 1-parameter family of SF couplings as response of the system to a change of a colour-electric background field. [Lüscher et al. '92]

A family of SF couplings II

- ν-dependence is explicit, obtained by computing $\bar{g}^2 \equiv \bar{g}^2_{\nu=0}$ and $\bar{v}$ at $\nu = 0$:

$$\frac{1}{\bar{g}^2_\nu} = \frac{1}{\bar{g}^2} - \nu \bar{v}$$

- relation between couplings at ν and $\nu = 0$ gives exact ratio:

$$r_\nu = \Lambda/\Lambda_\nu = \exp(-\nu \times 1.25516)$$

- The β-function is known to 3-loops:

$$(4\pi)^3 \times b_{2,\nu} = -0.06(3) - \nu \times 1.26$$

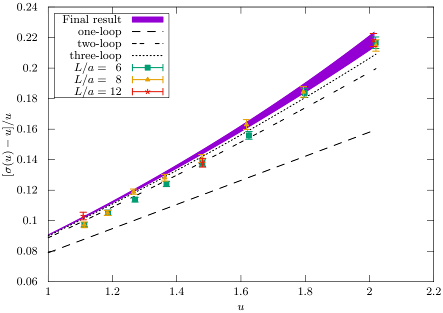
  N.B.: values ν of O(1) look perfectly fine!

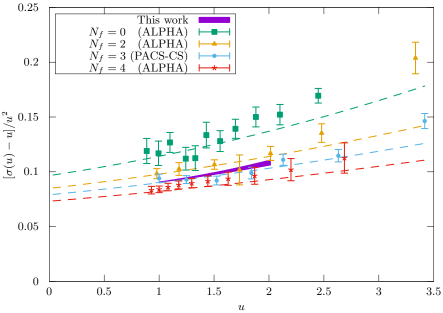
- infrared cutoff (finite volume) $\Rightarrow$ no renormalons; secondary minimum of the action:

$$\exp(-2.62/\alpha) \simeq (\Lambda/\mu)^{3.8}$$

- Cutoff effects: O(a^4) at tree-level, but O(a) effects from the boundaries:
  - subtracted perturbatively
  - variation of coefficients treated as systematic error, continuum extrapolations $\propto a^2$

SSF in the continuum limit





$\Rightarrow$ Significantly improved precision compared to previous work with $N_\mathrm{f} = 0, 2, 3, 4$

Computation of $L_0\Lambda$

- Define L_0 implicitly by
$$\bar{g}^2(L_0) = 2.012 = u_0$$

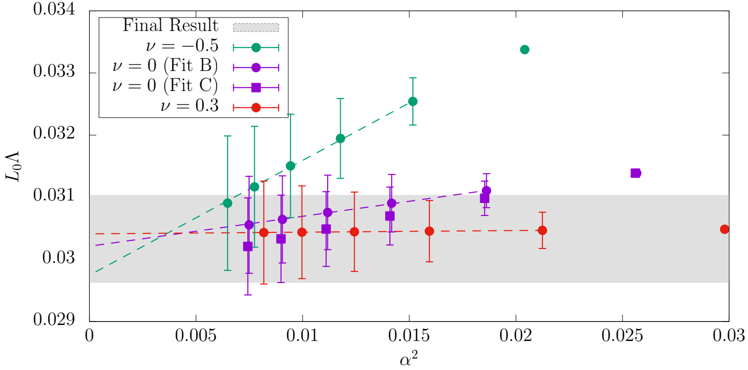
- Use the non-perturbative continuum step scaling function $\sigma(u)$:
$$u_{n-1} = \sigma(u_n), \quad n = 1, \ldots, \qquad \Rightarrow \quad u_n = \bar{g}^2\left(2^{-n} L_0\right)$$

- At scale $2^{-n}L_0$ obtain $L_0\Lambda$ using the perturbative β-function:
$$\begin{aligned} L_0\Lambda \quad &= \quad 2^n \left[b_0 \bar{g}^2(2^{-n}L_0)\right]^{-\frac{b_1}{2b_0^2}} \mathrm{e}^{-\frac{1}{2b_0 \bar{g}^2(2^{-n}L_0)}} \\ &\times \exp\left\{-\int_0^{\bar{g}(2^{-n}L_0)} dg \left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g}\right]\right\} \end{aligned}$$

- Do the same for schemes $\nu \neq 0$ using the continuum relation:
$$\frac{1}{\bar{g}_\nu^2(L_0)} = \frac{1}{2.012} - \nu \times 0.1199(10)$$

$\Rightarrow$ check accuracy of perturbation theory: $L_0\Lambda$ must be independent of ν and number of steps, n !

Result for $L_0\Lambda$



- All results agree around $\alpha = 0.1$, we quote

$$L_0\Lambda = 0.0303(7) \quad \Rightarrow \quad L_0\Lambda_{\overline{\text{MS}}}^{N_f=3} = 0.0791(19) \qquad \text{(error } 2.4\%)$$

Recall $L_0 \equiv L_{\text{swi}}$ is defined implicitly by $\bar{g}^2(L_0) = 2.012$.

Alternative test via the $\overline{\mathrm{MS}}$-scheme I

Idea: Perturbatively match the SF coupling to the $\overline{\mathrm{MS}}$-coupling then evaluate the Λ-parameter using the 5-loop β-function

- Relation between couplings, allowing for a scale factor s:

$$4\pi\alpha_{\overline{\mathrm{MS}}}(s/L) \equiv \bar{g}^2_{\overline{\mathrm{MS}}}(L/s) = \bar{g}^2_\nu(L) + p_1^\nu(s)\bar{g}^4_\nu(L) + p_2^\nu(s)\bar{g}^6_\nu(L) + \mathrm{O}(\bar{g}^8)$$

- Same as earlier, except now in the $\overline{\mathrm{MS}}$ scheme:

$$\Lambda_{\overline{\mathrm{MS}}} L_0 = \frac{sL_0}{L}\varphi_{\overline{\mathrm{MS}}}\left[\bar{g}_{\overline{\mathrm{MS}}}(L/s)\right] = s\, 2^n \varphi_{\overline{\mathrm{MS}}}\left[\sqrt{\bar{g}^2_\nu(L) + p_1^\nu(s)\bar{g}^4_\nu(L) + p_2^\nu(s)\bar{g}^6_\nu(L)}\right],$$

- expect to see independence of the number of steps n, scale factor s and parameter ν.
- Look at $\nu = 0$, depdendence on n and s.
- Note: The neglected order for Λ:

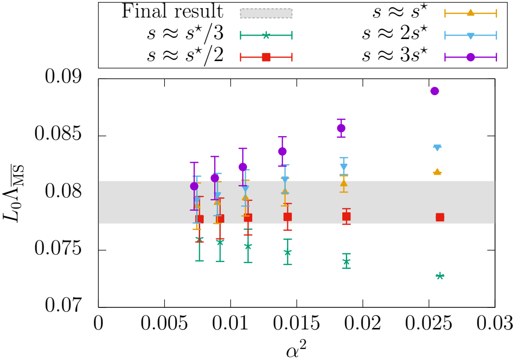
$$\Delta g^2 \frac{d\varphi}{dg^2} \propto \Delta g^2 \left\{g\beta(g)\right\}^{-1} = \Delta g^2 \times \mathrm{O}(g^{-4})$$

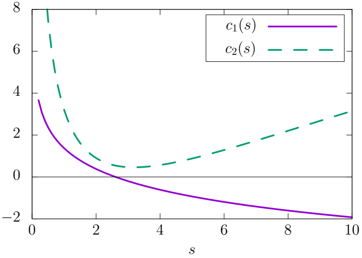
$\Rightarrow$ truncation error: $\mathrm{O}(g^8) \times \mathrm{O}(g^{-4}) = \mathrm{O}(g^4) = \mathrm{O}(\alpha^2)$.

Alternative test via the $\overline{\mathrm{MS}}$-scheme II

$$\alpha(sq) = \alpha_\nu(q) + c_1^\nu(s)\alpha_\nu^2 + c_2^\nu(s)\alpha_\nu^3(q) + \ldots, \qquad p_i^\nu = c_i^\nu/(4\pi)^i$$

parameters: $\nu = 0,\ s^* \approx 3$



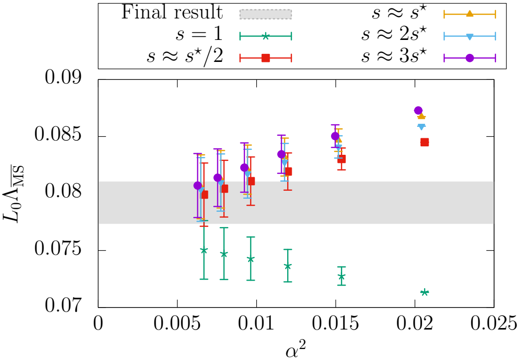


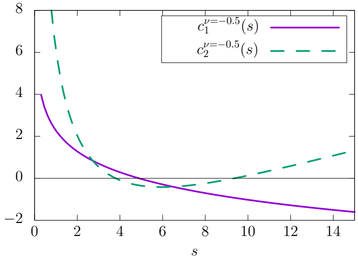
- Choice of scale factor is important, coefficients can get large.
- "fastest apparent convergence" principle: $c_1(s^*) = 0$ which means $s^* = \Lambda_{\overline{\mathrm{MS}}}/\Lambda = 2.612 \approx 3$ seems like a good idea.

Alternative test via the $\overline{\mathrm{MS}}$-scheme III

$$\alpha(sq) = \alpha_\nu(q) + c_1^\nu(s)\alpha_\nu^2 + c_2^\nu(s)\alpha_\nu^3(q) + \ldots, \qquad p_i^\nu = c_i^\nu/(4\pi)^i$$

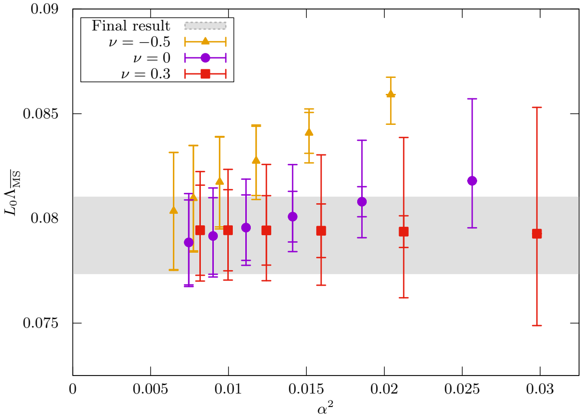
parameters: $\nu = -0.5$, $s^* \approx 5$





Alternative test via the $\overline{\text{MS}}$-scheme IV

variation of the scale factor $s \in [s^*/2, 2s^*]$



$\Rightarrow$ may significantly underestimate the systematic error!

Summary, tests of perturbation theory

- The determination of α_s is well-suited for the lattice approach;
- The systematics can be well controlled by combining technical tools developed over the last 25 years:
  - finite volume renormalization schemes and recursive step-scaling methods
  - non-perturbative Symanzik improvement
  - perturbation theory adapted to finite volume; relation between SF and $\overline{\mathrm{MS}}$-coupling known to 2-loop order!

$\Rightarrow$ Completely solves the problem of large scale differences;
perturbation theory at low energies can be avoided!

- Turning this around: many opportunities to test perturbation theory at high energies!

$\Rightarrow$ with hindsight: estimates of perturbative truncation errors require some luck!