Linear σ-model: $\mathcal{L}_\sigma = \frac{1}{2}\partial_\mu \phi \cdot \partial^\mu \phi - \frac{g}{4}(\phi^2 - v^2)^2$, $\phi^T = (\phi_0, \phi_1, \phi_2, \phi_3)$

SO(4) symmetry: $\phi \longrightarrow \phi' = R\phi$, $R = e^{\omega \varepsilon}$, $\varepsilon^T = -\varepsilon$, $\omega \in \mathbb{R}$

$$\approx (1+\varepsilon)\phi = \phi + \omega \underbrace{\delta\phi}, \quad \text{drop } O(\omega^2)$$

$$= \varepsilon\phi, \quad \delta\phi_i = \varepsilon_{ij}\phi_j$$

Six generators:

$$\begin{array}{ccc}
\epsilon_A^1 & \epsilon_A^2 & \epsilon_A^3 \\
\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
\epsilon_V^1 & \epsilon_V^2 & \epsilon_V^3
\end{array}$$

$\Rightarrow$ $$\varepsilon = \sum_i c_i \varepsilon_V^i + d_i \varepsilon_A^i$$

Noether currents (six, for each generator one)

$$j_i^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i$$

Explicit examples:

1) $\varepsilon = \varepsilon_V^1$: $\quad j_{V1}^\mu = -\partial^\mu \phi_2 \phi_3 + \partial^\mu \phi_3 \phi_2 = \varepsilon_{1bc}\phi_b \partial^\mu \phi_c$

2) $\varepsilon = \varepsilon_A^1$: $\quad j_{A1}^\mu = -\partial^\mu \phi_0 \phi_1 + \partial^\mu \phi_1 \phi_0$

$\Rightarrow$ conserved charge for 2): $\quad Q_{A^1} = \int d^3x \; -\dot{\phi}_0 \phi_1 + \dot{\phi}_1 \phi_0$

Conjugate momentum Π_j: $\quad = \int d^3x \; \Pi_1 \phi_0 - \Pi_0 \phi_1$

Spont. Sym Breaking for $v^2>0$

Assume groundstate $\phi_G^T = (v_0, \vec{0})$

Note: this is a particular choice. Rotate if not.

For the Goldstone theorem consider: $O = \phi_1$ with Q_{A1}:

$$\Rightarrow \quad [Q_{A1}, \phi_1] = \int d^3x\, [\pi_1 \phi_0 - \pi_0 \phi_1, \phi_1]$$

$$= \int d^3x\, \underbrace{[\pi_1, \phi_1]}_{-i\,\delta^{(3)}} \phi_0 = -i\,\phi_0$$

$$\Rightarrow \quad \langle 0 | [Q_{A1}, \phi_1] | 0 \rangle = -i \, \langle 0 | \phi_0 | 0 \rangle = -i\, v \neq 0$$

$\Rightarrow$ Goldstone theorem applies: NGB if $v^2>0$

Comment Noether theorem:

Consider local trafo: $R = R(x) = e^{\omega(x)\,\varepsilon} = 1 + \omega(x)\,\varepsilon$

$$\Rightarrow \quad \delta\mathcal{L} = \partial^\mu\phi\,(\partial_\mu\omega)\,\varepsilon\phi = \partial_\mu\omega\,\underbrace{(\partial^\mu\phi\,\varepsilon\,\phi)}_{j^\mu_{Noether}}$$

$j^\mu_{Noether}$, same as before!

a) Derive Vector current conservation

$$\mathcal{L}^0_{QCD} = \bar{q}\, i \gamma^\mu \partial_\mu q$$ Note: drop color part, invariant!

Vector trafo: $q' = V q \quad , \quad V = e^{i\omega} \quad , \quad \omega = \omega^a_{(x)} T^a$

$\bar{q}' = \bar{q} V^{-1} \qquad \simeq 1 + i\omega$

$$\Rightarrow \quad q' = q + \delta q \quad , \quad \delta q = i \omega^a T^a q$$

$$\bar{q}' = \bar{q} + \delta\bar{q} \quad , \quad \delta\bar{q} = -i\omega^a \bar{q} T^a$$

$$\Rightarrow \quad \delta \mathcal{L}^0_{QCD} = \bar{q}\, i\gamma^\mu \partial_\mu \delta q + \delta\bar{q}\, i\gamma^\mu \partial_\mu q$$

$$= \bar{q}\, i\gamma^\mu (i\partial_\mu \omega^a) T^a q + \underbrace{i\omega^a \bar{q}\, i\gamma^\mu T^a \partial_\mu q - i\omega^a \bar{q} T^a i\gamma^\mu \partial_\mu q}_{=0}$$

$$= -\partial_\mu \omega^a \left(\bar{q}\gamma^\mu T^a q\right) = -\partial_\mu \omega^a V^{\mu,a}$$

$$\Rightarrow \quad \delta S^0_{QCD} = \int d^4x\, \delta\mathcal{L}^0_{QCD} \overset{\text{p. Int.}}{=} \int d^4x\, \omega^a_{(x)} \partial_\mu V^{a,\mu}_{(x)}$$

if $O_{(y)} = V^{\nu,b}_{(y)} \quad , \quad y \notin R \Rightarrow \delta O = 0$

$$\Rightarrow \quad \boxed{\langle 0 | T\, \partial_\mu V^{\mu,a}_{(x)} V^{\nu,b}_{(y)} | 0 \rangle = 0}$$

b) $\mathcal{L}_{QCD} = \mathcal{L}^0_{QCD} + \mathcal{L}^m_{QCD}$

$$\mathcal{L}^m_{QCD} = -\bar{q}\mathcal{M}q \quad , \quad \mathcal{M} = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$$

$\Rightarrow$ consider vector trafo :

$$\delta\mathcal{L}^m_{QCD} = -\delta\bar{q}\mathcal{M}q - \bar{q}\mathcal{M}\delta q$$

$$= i\omega^a\bar{q}T^a\mathcal{M}q - i\omega^a\bar{q}\mathcal{M}T^aq$$

$$= -i\omega^a\bar{q}[\mathcal{M},T^a]q$$

$$\Rightarrow \delta S_{QCD} = \int d^4x \; \omega^a(x)\left(\partial_\mu V^{\mu,a}(x) - i\bar{q}[\mathcal{M},T^a]q\right)$$

$$\Rightarrow \boxed{\partial_\mu V^{\mu,a}(x) = i\bar{q}[\mathcal{M},T^a]q}$$

$\Rightarrow$ consider axial trafo

$$\Rightarrow \delta\mathcal{L}^m_{QCD} = -i\omega^a\bar{q}\gamma_5T^a\mathcal{M}q - i\omega^a\bar{q}\mathcal{M}\gamma_5T^aq$$

$$= -i\omega^a\bar{q}\{\mathcal{M},T^a\}\gamma_5q$$

$$\Rightarrow \delta S_{QCD} = \int d^4x \; \omega^a(x)\left(\partial_\mu A^{\mu,a}(x) - i\bar{q}\{\mathcal{M},T^a\}\gamma_5q\right)$$

$$\Rightarrow \boxed{\partial_\mu A^{\mu,a}(x) = i\bar{q}\{\mathcal{M},T^a\}\gamma_5q}$$

Consider $T^a = T^1 = \frac{\lambda^1}{2} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow \{\mathcal{M},T^1\} = (m_u+m_d)T^1$$

$$\Rightarrow \boxed{\partial_\mu A^{\mu,1}(x) = i(m_u+m_d)\bar{q}\gamma_5T^1q}$$